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The simple layer potential for the biharmonic equation in n variables

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Analisi matematica. — *The simple layer potential for the biharmonic equation in* n *variables.* Nota di ALBERTO CIALDEA, presentata (*) dal Socio G. FICHERA.

ABSTRACT. — A theory of the «simple layer potential» for the classical biharmonic problem in $\mathbf{R}^n$ is worked out. This hinges on the study of a new class of singular integral operators, each of them trasforming a vector with n scalar components into a vector whose components are n differential forms of degree one.

KEY WORDS: Singular integral operators; Differential forms; Biharmonic problem.

RIASSUNTO. — *Il potenziale di semplice strato per l'equazione biarmonica in* n *variabili.* Viene elaborata una teoria del «potenziale di semplice strato» per il classico problema biarmonico in $\mathbf{R}^n$. Essa è fondata sullo studio di una nuova classe di operatori integrali singolari ciascuno dei quali trasforma un vettore con n componenti scalari in un vettore avente come componenti n forme differenziali di grado uno.

1. INTRODUCTION

One of the consequences of Muskhelishvili's theory on singular integral equations is the solution of the Dirichlet problem for Laplace equation in two variables by means of a single layer potential. As it is known, if we try to solve the Dirichlet problem by means of such a potential we obtain an integral equation of the first kind on the curve Σ:

$$\int_{\Sigma} \varphi(y) \log|x-y|\, ds_y = g(x), \qquad x \in \Sigma. \tag{1.1}$$

Differentiating both sides of (1.1) with respect to the arc-length we obtain a singular integral equation that can be reduced to a Fredholm equation (see [5, 12]). The method of Muskhelishvili was extended by G. Fichera to elliptic equations of higher order in two variables (see [7]). In [1] the solution of the Dirichlet problem for Laplace equation *in n variables* by means of a single layer potential is obtained in the following way. Boundary condition leads to this integral equation of the first kind on the boundary Σ:

$$\int_{\Sigma} \varphi(y) |x-y|^{2-n}\, d\sigma_y = g(x), \qquad x \in \Sigma.$$

If we consider *the differential* of both sides we obtain the following singular integral equation:

$$\int_{\Sigma} \varphi(y)\, d_x [|x-y|^{2-n}]\, d\sigma_y = dg(x), \qquad x \in \Sigma. \tag{1.2}$$

In this equation the unknown is a scalar function, but *the data is a differential*

(*) Nella seduta del 10 novembre 1990.

form. In [1] it was shown that the operator on the left-hand side can be reduced and so we can apply the general theory of such operators (see [8]). In this way we obtain, for example, that there exists a solution $\varphi \in L^p(\Sigma)$ $(C^\lambda(\Sigma))$ of (1.2) if and only if $g \in W^{1,p}(\Sigma)$ $(C^{1+\lambda}(\Sigma))$. Then we can represent the solution of the Dirichlet problem by means of a single layer potential with a density belonging to $L^p(\Sigma)$ $(C^\lambda(\Sigma))$ if and only if the data g belongs to $W^{1,p}(\Sigma)$ $(C^{1+\lambda}(\Sigma))$. In this paper we consider the Dirichlet problem for the biharmonic equation in n variables $(n \geq 3)$. The following integral:

$$u(x) = \int_\Sigma \varphi_h(y) \frac{\partial}{\partial y_h} F(x,y)\, d\sigma_y \tag{1.3}$$

(where $F(x,y)$ is the fundamental solution for the biharmonic operator) will be called a *biharmonic simple layer potential.* The boundary conditions $u|_\Sigma = g_0$, $(\partial u/\partial x_k)|_\Sigma = g_k$ $(k = 1, \dots, n)$ lead to a system of integral equations of the first kind. If we consider the differentials $d(\partial u/\partial x_h)$ we obtain the singular integral system

$$\int_\Sigma \varphi_h(y)\, d_x\left[\frac{\partial^2}{\partial x_k \partial y_h} F(x,y)\right] d\sigma_y = dg_k(x), \qquad x \in \Sigma \quad (k = 1, \dots, n).$$

This system appears to be analogous to the equation (1.2), but a difficulty arising: the operator on the left-hand side *cannot be reduced* (if $n \geq 3$). We overcome this difficulty by constructing another operator having the same range, but such that it can be reduced. In this way we shall be able to prove an existence and uniqueness theorem for the Dirichlet problem for the biharmonic equation in the class $C^{2+\lambda}(\overline{\Omega}) \cap C^\omega(\Omega)$. More important, the method provides the representation of the solution through the *simple layer potential* (1.3).

2. Definitions and notations

Let B and B' two Banach spaces. Let S: $B \to B'$ be a linear and continuous operator. We say that S *can be reduced* if there exists another operator S': $B' \to B$ such that $S'S = I + \mathcal{T}$, where I is the identity and $\mathcal{T}: B \to B$ is a completely continuous operator.

If S can be reduced then:

i) the dimension of the kernel $\mathcal{N}(S)$ is finite;

ii) the range $\mathcal{R}(S)$ is closed in B';

iii) there exists a solution $\varphi \in B$ of the equation $S\varphi = \psi$ $(\psi \in B')$ if and only if $\langle \gamma, \psi \rangle = 0$ for any $\gamma \in B'^*$ such that $S^* \gamma = 0$ (B'^* is the topological dual space of B' and S^*: $B'^* \to B^*$ is the adjoint of S). For the proofs of these theorems see [8]. We remark that the dimension of $\mathcal{N}(S^*)$ *can* be infinte.

Let now Ω be a bounded domain of $\mathbf{R}^n$, such that $\Sigma = \partial\Omega$ is a Lyapunov boundary. This means that Σ has a uniformly Hölder continuous normal field of some exponent μ $(0 < \mu \leq 1)$.

Let $1<p<\infty$: we denote by $L^p(\Omega)$ ($L^p(\Sigma)$) the vector space of all measurable real functions such that $|u|^p$ is integrable over Ω (over Σ).

$L^p_k(\Sigma)$ is the vector space of the differential forms of degree k defined on Σ such that its components are integrable functions belonging to $L^p(\Sigma)$ in a coordinate system of class C^1 and consequently in every coordinate system of class C^1 (see [6]. A summary on k-forms can be found in [2], section 4).

We denote by $W^{1,p}(\Sigma)$ the vector space of the functions u belonging to $L^p(\Sigma)$ such that the weak differential du belongs to $L^p_1(\Sigma)$.

$C^\lambda(\overline{\Omega})$ ($C^\lambda(\Sigma)$) will denote the vector space of all continuous functions satisfying in $\overline{\Omega}$ (on Σ) a uniform Hölder condition of some exponent λ $(0<\lambda\leq 1)$.

By $C^{k+\lambda}(\overline{\Omega})$ we shall denote the sub-class of $C^k(\overline{\Omega})$ consisting of functions u such that $D^\alpha u\in C^\lambda(\overline{\Omega})$ $(|\alpha|=k)$.

$C^\omega(\Omega)$ is the space of analytic functions defined in Ω.

Let $s_0(x,y)$ be the fundamental solution for Laplace equation:

$$s_0(x,y)=\begin{cases}(2\pi)^{-1}\log|x-y| & n=2\\ -[(n-2)\,c_n]^{-1}\,|x-y|^{2-n} & n=3,4,\dots\end{cases}$$

where c_n is the hypersurface measure of the unit sphere of $\mathbf{R}^n$. Let $s_k(x,y)$ be the Hodge form:

$$\sum_{j_1<\dots<j_k} s_0(x,y)\,dx^{j_1}\dots dx^{j_k}dy^{j_1}\dots dy^{j_k}.$$

Let us consider the following operator:

$$J\colon L^p(\Sigma)\to L^p_1(\Sigma),\qquad J\varphi(x)=\int_\Sigma \varphi(y)\,d_x[s_0(x,y)]\,d\sigma_y,$$

where d_x is the differential operator acting on the variables $(x_1,\dots,x_n)$. It is worthwhile to remark that the coefficients of the form on the right-hand side are *singular integrals* (see [1, 3, 10, 11]). In [1] it is proved that J can be reduced, namely

$$J'J\varphi(z)=-\frac{1}{4}\varphi(z)+\int_\Sigma\varphi(y)\,L(z,y)\,d\sigma_y,\qquad \forall\varphi\in L^p(\Sigma)\tag{2.1}$$

where $L(z,y)$ is a kernel with a weak singularity and:

$$J'\colon L^p_1(\Sigma)\to L^p(\Sigma),\qquad J'\psi(z)=*_\Sigma\int_\Sigma\psi(x)\wedge d_z[s_{n-2}(z,x)]$$

($*_\Sigma$ has the following meaning: if γ is a $(n-1)$-form on Σ, say $\gamma=\gamma_0\,d\sigma$, γ_0 being a scalar function, then $*_\Sigma\gamma=\gamma_0$). It will be useful to write the left-hand side of (2.1) in the following way. Let us introduce the tangential operators $M^{j_1\dots j_{n-2}}u=$ $=*_\Sigma(du\wedge dx^{j_1}\dots dx^{j_{n-2}})$. It is not difficult to see that we can write:

$$J'J\varphi(z)=\sum_{j_1<\dots<j_{n-2}}\int_\Sigma M_z^{j_1\dots j_{n-2}}[s_0(z,x)]\,d\sigma_x\int_\Sigma M_x^{j_1\dots j_{n-2}}[s_0(x,y)]\,\varphi(y)\,d\sigma_y.\tag{2.2}$$

Let us consider the fundamental solution for the biharmonic equation:

$$F(x,y)=\begin{cases}[c_n(n-2)(n-4)]^{-1}|x-y|^{4-n} & n=3,5,6,\dots\\ -(2c_4)^{-1}\log|x-y| & n=4.\end{cases}$$

We observe that $\partial F(x,y)/\partial x_k=(x_k-y_k)\,s_0(x,y)$; $\Delta_2 F(x,y)=2\,s_0(x,y)$ $(n=3,4,5,6,\dots)$; $\Delta_2=\sum_{k=1}^{n}\partial^2/\partial x_k^2$.

The following integral will be called a *biharmonic simple layer potential:*

(2.3) $$u(x)=\int_\Sigma \varphi_h(y)\frac{\partial}{\partial y_h}F(x,y)\,d\sigma_y\,.$$

Obviously we can write:

$$\frac{\partial}{\partial x_k}u(x)=\int_\Sigma \varphi_h(y)\frac{\partial^2}{\partial x_k\,\partial y_h}F(x,y)\,d\sigma_y\,.$$

Moreover arguing as in [10] (pp. 310-312) it is possible to show that:

$$\lim_{x'\to x}\frac{\partial^2}{\partial x_j\partial x_k}u(x')=c\,\nu_j(x)\,\nu_k(x)\,\nu_h(x)\,\varphi_h(x)+\int_\Sigma\varphi_h(y)\frac{\partial^3}{\partial x_j\partial x_k\partial y_h}F(x,y)\,d\sigma_y\,,\quad a.e.\ x\in\Sigma$$

where the limit denotes the internal angular boundary value (see [10], p. 239), $(\nu_1,\dots,\nu_n)$ is the outward unit normal to Σ and c is a constant we do not need to write explicitly. Keeping in mind that

(2.4) $$\nu_j(x)\,dx^j=0\quad \text{on } \Sigma$$

we have

(2.5) $$\lim_{x'\to x}d\left(\frac{\partial}{\partial x_k}u(x')\right)=\int_\Sigma\varphi_h(y)\,d_x\left[\frac{\partial^2}{\partial x_k\partial y_h}F(x,y)\right]d\sigma_y\,,\qquad a.e.\ x\in\Sigma\,.$$

3. Study of a particular operator

I. *Let $v(x)$ be the following function:*

(3.1) $$v(x)=\int_\Sigma\psi(y)\frac{\partial}{\partial\nu_y}F(x,y)\,d\sigma_y+$$
$$+\sum_{j_1<\dots<j_{n-2}}\int_\Sigma\Phi(y)\wedge M_y^{j_1\dots j_{n-2}}[F(x,y)]\,dy^{j_1}\dots dy^{j_{n-2}}$$

where $\psi\in L^p(\Sigma)$, $\Phi\in L_1^p(\Sigma)$. We have that:

(3.2) $$*_\Sigma\,\nu_k(z)\int_\Sigma d\frac{\partial v}{\partial x_k}\wedge d_z[s_{n-2}(z,x)]=\frac{1}{4}\psi+\mathcal{T}_1\psi+\mathcal{T}_2\Phi$$

where the operators $\mathcal{T}_1$: $L^p(\Sigma)\to L^p(\Sigma)$, $\mathcal{T}_2$: $L_1^p(\Sigma)\to L^p(\Sigma)$ are completely continuous.

Set

$$I_k(x) = \int_\Sigma \psi(y) \frac{\partial}{\partial \nu_y} \frac{\partial}{\partial x_k} F(x,y)\, d\sigma_y ;$$

$$J_k(x) = \sum_{j_1 < \ldots < j_{n-2}} \int_\Sigma \Phi(y) \wedge M_y^{j_1 \ldots j_{n-2}} \left[\frac{\partial}{\partial x_k} F(x,y) \right] dy^{j_1} \ldots dy^{j_{n-2}} .$$

We can write: $\partial v(x)/\partial x_k = I_k(x) + J_k(x)$. Since

$$I_k(x) = \int_\Sigma \psi(y)(x_k - y_k) \frac{\partial}{\partial \nu_y} s_0(x,y)\, d\sigma_y - \int_\Sigma \psi(y)\, \nu_k(y)\, s_0(x,y)\, d\sigma_y$$

we have:

$$\nu_k(z) \int_\Sigma dI_k(x) \wedge d_z [s_{n-2}(z,x)] =$$

$$= \int_\Sigma [\nu_k(z) - \nu_k(x)] \int_\Sigma \psi(y)\, d_x [(x_k - y_k) \frac{\partial}{\partial \nu_y} s_0(x,y)]\, d\sigma_y \wedge d_z [s_{n-2}(z,x)] +$$

$$+ \int_\Sigma \int_\Sigma \psi(y) \frac{\partial}{\partial \nu_y} s_0(x,y)\, d\sigma_y\, \nu_k(x)\, dx^k \wedge d_z [s_{n-2}(z,x)] +$$

$$+ \int_\Sigma \int_\Sigma \psi(y)\, \nu_k(x)(x_k - y_k)\, d_x \left[\frac{\partial}{\partial \nu_y} s_0(x,y) \right] d\sigma_y \wedge d_z [s_{n-2}(z,x)] -$$

$$- \nu_k(z) \int_\Sigma \int_\Sigma \psi(y)\, \nu_k(y)\, d_x [s_0(x,y)]\, d\sigma_y \wedge d_z [s_{n-2}(z,x)] .$$

Using the fact that:

$$\nu_k(z) - \nu_k(x) = \mathcal{O}(|z-x|^\mu) \tag{3.3}$$

$$\nu_k(x)(x_k - y_k) = \mathcal{O}(|x-y|^{1+\mu}) \tag{3.4}$$

and applying (2.1), (2.4) we obtain:

$$*_\Sigma\, \nu_k(z) \int_\Sigma dI_k(x) \wedge d_z [s_{n-2}(z,x)] = \nu_k(z) \left[\frac{1}{4} \psi(z)\, \nu_k(z) + \tilde{\mathcal{T}}_k \psi \right] = \frac{1}{4} \psi + \mathcal{T}_1 \psi$$

where $\mathcal{T}_1 = \nu_k \tilde{\mathcal{T}}_k$ is a completely continuous operator from $L^p(\Sigma)$ into itself. On the other hand

$$\nu_k(z) \int_\Sigma dJ_k(x) \wedge d_z [s_{n-2}(z,x)] = \int_\Sigma [\nu_k(z) - \nu_k(x)]\, dJ_k(x) \wedge d_z [s_{n-2}(z,x)] +$$

$$+ \int_\Sigma \nu_k(x)\, dJ_k(x) \wedge d_z [s_{n-2}(z,x)] .$$

Moreover:

$$\nu_k(x)\, dJ_k(x) = \sum_{j_1<\ldots<j_{n-2}} \nu_k(x) \left\{ \int_\Sigma \Phi(y) \wedge M_y^{j_1\ldots j_{n-2}}[s_0(x,y)]\, dy^{j_1}\ldots dy^{j_{n-2}} dx^k + \right.$$

$$+ \int_\Sigma \Phi(y) \wedge (x_k - y_k)\, d_x\{M_y^{j_1\ldots j_{n-2}}[s_0(x,y)]\}\, dy^{j_1}\ldots dy^{j_{n-2}} -$$

$$\left. - \int_\Sigma \Phi(y) \wedge M^{j_1\ldots j_{n-2}}(y_k)\, d_x[s_0(x,y)]\, dy^{j_1}\ldots dy^{j_{n-2}} \right\}.$$

Since we have: $\nu_k(y)\; M^{j_1\ldots j_{n-2}}(y_k) = 0$, we can write: $\nu_k(x)\; M^{j_1\ldots j_{n-2}}(y_k) = [\nu_k(x) - -\nu_k(y)]\, M^{j_1\ldots j_{n-2}}(y_k) = \mathcal{O}(|x-y|^\mu)$ and hence, with the aid of (2.4), (3.4)

$$\nu_k(z) \int_\Sigma dJ_k(x) \wedge d_z[s_{n-2}(z,x)] = \mathcal{T}_2 \Phi$$

where $\mathcal{T}_2$ is completely continuous from $L_1^p(\Sigma)$ into $L^p(\Sigma)$.

If we suppose something more, namely that the form Φ is the differential of a (harmonic) simple layer potential, *i.e.*

$$\Phi(y) = \int_\Sigma \varphi(w)\, d_y[s_0(y,w)]\, d\sigma_w \qquad (\varphi \in L^p(\Sigma)) \tag{3.5}$$

we can prove the following result:

II. *Let* $v(x)$ *be the function defined by* (3.1) *where* Φ *is given by* (3.5) *We have:*

$$*_\Sigma \left[\frac{1}{(n-2)!}\, \delta^{1\ldots n}_{i_1\ldots i_n}\, \nu_{i_1}\, d\frac{\partial v}{\partial x_{i_2}} \wedge dx^{i_3}\ldots dx^{i_n} \right] = -\frac{1}{2}\varphi + \mathcal{T}_3\psi + \mathcal{T}_4\varphi \tag{3.6}$$

where $\mathcal{T}_3$, $\mathcal{T}_4$: $L^p(\Sigma) \to L^p(\Sigma)$ *are completely continuous operators.*

First of all note that the left-hand side in (3.6) is equal to

$$\Delta_2 v - \nu_h \frac{\partial}{\partial \nu}\left(\frac{\partial v}{\partial x_h}\right).$$

In fact we have:

$$*_\Sigma \left[\frac{1}{(n-2)!}\, \delta^{1\ldots n}_{i_1\ldots i_n}\, \nu_{i_1}\, d\frac{\partial v}{\partial x_{i_2}} \wedge dx^{i_3}\ldots dx^{i_n} \right] =$$

$$= *_\Sigma \left[\frac{1}{(n-2)!}\, \delta^{1\ldots n}_{i_1\ldots i_n}\, \nu_{i_1} \frac{\partial^2 v}{\partial x_k \partial x_{i_2}}\, dx^k dx^{i_3}\ldots dx^{i_n} \right] =$$

$$= \frac{1}{(n-2)!}\, \delta^{1\ldots n}_{i_1\ldots i_n}\, \delta^{1\ldots\ldots n}_{jki_3\ldots i_n}\, \nu_{i_1}\nu_j \frac{\partial^2 v}{\partial x_k \partial x_{i_2}} = \delta^{j\,k}_{i_1 i_2}\, \nu_{i_1}\nu_j \frac{\partial^2 v}{\partial x_k \partial x_{i_2}} = \Delta_2 v - \nu_h \frac{\partial}{\partial \nu}\left(\frac{\partial v}{\partial x_h}\right).$$

Then with the aid of (2.5) we can write [1]

$$\Delta_2 v - \nu_h \frac{\partial}{\partial \nu}\left(\frac{\partial v}{\partial x_h}\right) = \int_\Sigma \psi(y) \frac{\partial}{\partial \nu_y}\left\{2 s_0(x,y) - \nu_h(x) \frac{\partial}{\partial \nu_x}[(x_h - y_h) s_0(x,y)]\right\} d\sigma_y +$$

$$+ \sum \int_\Sigma \Phi(y) \wedge M_y^{j_1 \dots j_{n-2}}\left\{2 s_0(x,y) - \nu_h(x) \frac{\partial}{\partial \nu_x}[(x_h - y_h) s_0(x,y)]\right\} dy^{j_1} \dots dy^{j_{n-2}}.$$

The first integral is completely continuous from $L^p(\Sigma)$ into itself; the second one can be rewritten in the following way:

$$\sum\left\{2 \int_\Sigma \Phi(y) \wedge M_y^{j_1 \dots j_{n-2}}[s_0(x,y)]\, dy^{j_1} \dots dy^{j_{n-2}} - \right.$$

$$- \nu_h(x) \nu_h(x) \int_\Sigma \Phi(y) \wedge M_y^{j_1 \dots j_{n-2}}[s_0(x,y)]\, dy^{j_1} \dots dy^{j_{n-2}} -$$

$$\left. - \int_\Sigma \Phi(y) \wedge M_y^{j_1 \dots j_{n-2}}\left\{\nu_h(x)(x_h - y_h) \frac{\partial}{\partial \nu_x}[s_0(x,y)]\right\} dy^{j_1} \dots dy^{j_{n-2}}\right\} = I(x) - J(x),$$

where $I(x)$ is the first integral and $J(x)$ denotes the sum of the other two terms. Because of (2.1), (2.2), (3.5) we have

$$I(x) = 2 \sum \int_\Sigma \int_\Sigma \varphi(w)\, d_y[s_0(y,w)]\, d\sigma_w \wedge M_y^{j_1 \dots j_{n-2}}[s_0(x,y)]\, dy^{j_1} \dots dy^{j_{n-2}} =$$

$$= 2 \sum \int_\Sigma M_y^{j_1 \dots j_{n-2}}[s_0(x,y)]\, d\sigma_y \int_\Sigma \varphi(w) M_y^{j_1 \dots j_{n-2}}[s_0(y,w)]\, d\sigma_w =$$

$$= 2 \sum \int_\Sigma (M_y^{j_1 \dots j_{n-2}} - M_x^{j_1 \dots j_{n-2}})[s_0(x,y)]\, d\sigma_y \int_\Sigma \varphi(w) M_y^{j_1 \dots j_{n-2}}[s_0(y,w)]\, d\sigma_w +$$

$$+ 2 \sum \int_\Sigma M_x^{j_1 \dots j_{n-2}}[s_0(x,y)]\, d\sigma_y \int_\Sigma \varphi(w) M_y^{j_1 \dots j_{n-2}}[s_0(y,w)]\, d\sigma_w = -\frac{1}{2}\varphi(x) + \mathcal{T}\varphi$$

where $\mathcal{T}$ is a completely continuous operator from $L^p(\Sigma)$ into itself [2].

From (3.4) it follows that $J(x) = K(\Phi)$, where $K\colon L_1^p(\Sigma) \to L^p(\Sigma)$ is completely continuous. Thus to conclude the proof of this theorem it is sufficient to observe that the integral on the right-hand side of (3.5) is a continuous operator from $L^p(\Sigma)$ into $L_1^p(\Sigma)$.

Let H be the following operator: $H\colon (\psi, \varphi) \in [L^p(\Sigma)]^2 \to (d(\partial v/\partial x_1), \dots, d(\partial v/\partial x_n)) \in [L_1^p(\Sigma)]^n$ where v is given by (3.1), (3.5).

(1) From now on we shall indicate the summation by Σ without $j_1 < \dots < j_{n-2}$.

(2) We observe that $(M_y^{j_1 \dots j_{n-2}} - M_x^{j_1 \dots j_{n-2}})\,[s_0(x,y)] = \mathcal{O}(|x-y|^{1-m+\mu})$.

III. *The operator H can be reduced. Then the range $\mathcal{R}(H)$ is closed in $[L_1^p(\Sigma)]^n$.*

This theorem follows immediately from Theorems I, II.

4. The simple layer potential

Let $u(x)$ be the simple layer potential (2.3). We consider the following operator: S: $(\varphi_1, \dots, \varphi_n) \in [L^p(\Sigma)]^n \to (d(\partial u/\partial x_1), \dots, d(\partial u/\partial x_n)) \in [L_1^p(\Sigma)]^n$. In other words the k-th component of $S(\varphi_1, \dots, \varphi_n)$ is given by:

$$\int_\Sigma \varphi_h(y)\, d_x \left[\frac{\partial^2}{\partial x_k \partial y_h} F(x,y) \right] d\sigma_y .$$

It is worthwhile to remark that this operator *cannot be reduced,* because there are infinite linearly independent $(\varphi_1, \dots, \varphi_n)$ such that $u(x) \equiv 0$, and so the dimension of $\mathcal{N}(S)$ is infinite. However the following theorem holds.

IV. $\mathcal{R}(S) = \mathcal{R}(H)$, *where H is the operator studied in the previous section.*

With calculations we have

$$\frac{\partial}{\partial x_h} = \nu_h \frac{\partial}{\partial \nu} - \delta^{1 \dots\dots n}_{h i_2 \dots i_n} \nu_{i_2} M^{i_3 \dots i_n} .$$

Then if u is given by (2.3) we can write

$$u(x) = \int_\Sigma \varphi_h(y)\, \nu_h(y) \frac{\partial}{\partial \nu_y} F(x,y)\, d\sigma_y - \delta^{1 \dots n}_{i_1 \dots i_n} \int_\Sigma \varphi_{i_1}(y)\, \nu_{i_2}(y)\, M_y^{i_3 \dots i_n} [F(x,y)]\, d\sigma_y .$$

On the other hand setting $\Phi(y) = \varphi_h(y)\, dy^h/(n-2)!$, we have

$$\sum \Phi(y) \wedge M_y^{j_1 \dots j_{n-2}} [F(x,y)]\, dy^{j_1} \dots dy^{j_{n-2}} = \varphi_h(y)\, M_y^{j_1 \dots j_{n-2}} [F(x,y)]\, dy^h\, dy^{j_1} \dots dy^{j_{n-2}} =$$
$$= \delta^{1 \dots\dots n}_{k h j_1 \dots j_{n-2}} \varphi_h(y)\, \nu_k(y)\, M_y^{j_1 \dots j_{n-2}} [F(x,y)]\, d\sigma_y .$$

Then

$$u(x) = \int_\Sigma \psi(y) \frac{\partial}{\partial \nu_y} F(x,y)\, d\sigma_y + \sum \int_\Sigma \Phi(y) \wedge M_y^{j_1 \dots j_{n-2}} [F(x,y)]\, dy^{j_1} \dots dy^{j_{n-2}} \tag{4.1}$$

where $\psi = \varphi_h \nu_h \in L^p(\Sigma)$ and $\Phi \in L_1^p(\Sigma)$. This means that if u is given by (2.3) then it can be written in the form (4.1). Conversely it is obvious that if u is given by (4.1) then it can be written in the form (2.3). From that it follows easily that $\mathcal{R}(H) \subseteq \mathcal{R}(S)$. To conclude the proof it is sufficient to show that for any $\Phi \in L_1^p(\Sigma)$ there exists $g \in L^p(\Sigma)$ such that

$$\sum \int_\Sigma \Phi(y) \wedge M_y^{j_1 \dots j_{n-2}} [F(x,y)]\, dy^{j_1} \dots dy^{j_{n-2}} =$$
$$= \sum \int_\Sigma G(y) \wedge M_y^{j_1 \dots j_{n-2}} [F(x,y)]\, dy^{j_1} \dots dy^{j_{n-2}} \qquad \forall x \in \mathbf{R}^n ,$$

where

$$G(y) = \int_{\Sigma} g(w)\, d_y\,[s_0(y, w)]\, d\sigma_w\,.$$

This is equivalent to prove that for any $\Phi \in L_1^p(\Sigma)$ there exists $g \in L^p(\Sigma)$ such that

$$\sum \int_{\Sigma} \Phi(y) \wedge M^{j_1 \cdots j_{n-2}}[f(y)]\, dy^{j_1} \ldots dy^{j_{n-2}} = \tag{4.2}$$

$$= \sum \int_{\Sigma} g(w)\, d\sigma_w \int_{\Sigma} M^{j_1 \cdots j_{n-2}}[f(y)]\, d_y\,[s_0(y, w)] \wedge dy^{j_1} \ldots dy^{j_{n-2}}\,, \qquad \forall f \in C^1(\Sigma)\,.$$

Set

$$M_1 f = \sum M^{j_1 \cdots j_{n-2}}[f(y)]\, dy^{j_1} \ldots dy^{j_{n-2}}\,,$$

$$M_2 f = \sum \int_{\Sigma} M^{j_1 \cdots j_{n-2}}[f(y)]\, d_y\,[s_0(y, w)] \wedge dy^{j_1} \ldots dy^{j_{n-2}}\,.$$

We have:

$$\|M_1 f\|_{L^p_{n-2}(\Sigma)} \leq C \|M_2 f\|_{L^p(\Sigma)}\,, \qquad \forall f \in C^1(\Sigma) \tag{4.3}$$

In fact, if $f \in W^{1,p}(\Sigma)$ there exists $\lambda \in L^p(\Sigma)$ such that (3)

$$f(y) = \int_{\Sigma} \lambda(t)\, s_0(y, t)\, d\sigma_t\,.$$

Arguing as in the proof of Theorem II it is possible to show that: $M_2 f = -\lambda/4 + \mathcal{T}\lambda$, where $\mathcal{T}$ is a completely continuous operator from $L^p(\Sigma)$ into itself. This implies the following inequality:

$$\inf_{\lambda_0 \in N} \|\lambda + \lambda_0\|_{L^p(\Sigma)} \leq C_1 \|M_2 f\|_{L^p(\Sigma)}\,, \qquad \forall \lambda \in L^p(\Sigma) \tag{4.4}$$

where N is the class of all the functions $\lambda_0 \in L^p(\Sigma)$ such that $M_2 f_0 = 0$,

$$f_0(y) = \int_{\Sigma} \lambda_0(t)\, s_0(y, t)\, d\sigma_t\,.$$

On the other hand $M_2 f_0 = 0$ if and only if (see [1], pp. 189-190)

$$\sum \int_{\Sigma} M^{j_1 \cdots j_{n-2}}[f_0(y)]\, dg(y) \wedge dy^{j_1} \ldots dy^{j_{n-2}} = 0\,, \qquad \forall g \in C^1(\Sigma)$$

i.e.

$$\sum \int_{\Sigma} M^{j_1 \cdots j_{n-2}} f_0\, M^{j_1 \cdots j_{n-2}} g\, d\sigma = 0\,, \qquad \forall g \in C^1(\Sigma)\,.$$

This is possible if and only if $M^{j_1 \cdots j_{n-2}} f_0 = 0$, *i.e.* $f_0 = \text{const.}$ on Σ and then in $\overline{\Omega}$.

(3) This can be easily obtained from the results contained in [1].

Then we can write:

(4.5) $$\|M_1 f\|_{L^p_{n-2}(\Sigma)} = \|M_1(f+f_0)\|_{L^p_{n-2}(\Sigma)} \leq C_2 \|\lambda + \lambda_0\|_{L^p(\Sigma)}, \qquad \forall \lambda_0 \in N.$$

From (4.4), (4.5) it follows (4.3). By using an existence principle of Functional Analysis (4) we obtain the existence of a solution g of (4.2).

V. *$\mathcal{R}(S)$ is closed in*$[L^p_1(\Sigma)]^n$.

It follows immediately from Theorems III, IV.

We conclude this section with the following existence theorem which is a consequence of the previous one.

VI. *Given the forms $\Gamma_h \in L^p_1(\Sigma)$ $(h = 1, \dots, n)$ there exists $(\varphi_1, \dots, \varphi_n) \in [L^p(\Sigma)]^n$ such that $d(\partial u/\partial x_h) = \Gamma_h$ $(h = 1, \dots, n)$, where u is given by* (2.3) *if and only if*

(4.6) $$\int_\Sigma \Gamma_h \wedge \zeta_h = 0$$

for any $(\zeta_1, \dots, \zeta_n) \in L^q_{n-2}(\Sigma)$ $(1/p + 1/q = 1)$ *such that* $S^*(\zeta_1, \dots, \zeta_n) = 0$, i.e.

(4.7) $$\int_\Sigma \zeta_h(x) \wedge d_x \left[\frac{\partial^2}{\partial x_k \partial x_h} F(x,y) \right] = 0 \qquad \text{a.e.} \quad y \in \Sigma, \quad (k = 1, \dots, n).$$

5. The compatibility conditions

In this section we want to determine the eigensolutions of (4.7). For semplicity we consider only the case $p = q = 2$. We say that $(\zeta_1, \dots, \zeta_n) \in L^2_{n-2}(\Sigma)$ is an *eigensolution of the first kind* of (4.7) when $(\zeta_1, \dots, \zeta_n) \neq 0$ and

(5.1) $$\int_\Sigma \zeta_h(x) \wedge d_x \left[\frac{\partial}{\partial x_h} F(x,y) \right] = 0 \qquad y \in \mathbf{R}^n.$$

An eigensolution that is not of the first kind is called of *the second kind*. If

(5.2) $$\zeta_h = x_h \, \alpha + \eta_h \qquad (h = 1, \dots, n).$$

where α, η_h are weakly closed forms belonging to $L^2_{n-2}(\Sigma)$, then $(\zeta_1, \dots, \zeta_n)$ is an eigensolution of the first kind. On the other hand, if $(\zeta_1, \dots \zeta_n)$ is an eigensolution of the first kind then it must be written in the form (5.2). In fact if $f(x)$ is an infinite differentiable function with a compact support, it can be represented in the following way:

$$f(x) = \int_{\mathbf{R}^n} \mu(y) \, F(x,y) \, dy.$$

(4) See the Lecture 2 of [9].

From (5.1) it follows that

$$\int_{\Sigma} \zeta_h \wedge d \frac{\partial f}{\partial x_h} = 0$$

for any test function f and so (5.2) holds.

Let now be $(\bar{\zeta}_1, \ldots, \bar{\zeta}_n)$ an eigensolution of the second kind. From (4.7) it follows that

$$\int_{\Sigma} \bar{\zeta}_h(x) \wedge d_x \left[\frac{\partial}{\partial x_h} F(x,y) \right] = c \qquad a.e. \quad y \in \Sigma \tag{5.3}$$

where c is a constant. Now we prove that $c \neq 0$. Let us suppose that $c=0$ and let $v(y)$ be the function on the left-hand side of (5.3).

Let $\{(\bar{\zeta}_1^{(m)}, \ldots, \bar{\zeta}_n^{(m)});\ \bar{\zeta}_h^{(m)} \in C^{\lambda}_{n-2}(\Sigma)\}$ be a sequence such that

$$\lim_{m\to\infty} \|\bar{\zeta}_h^{(m)} - \bar{\zeta}_h\|_{L^2_{n-2}(\Sigma)} = 0 \qquad (h = 1, \ldots, n)$$

and let $v^{(m)}$ be the following function:

$$v^{(m)}(y) = \int_{\Sigma} \bar{\zeta}_h^{(m)}(x) \wedge d_x \left[\frac{\partial}{\partial x_h} F(x,y) \right].$$

These biharmonic functions are smooth and such that:

$$\lim_{m\to\infty} \|v^{(m)}\|_{L^2(\Sigma)} = \lim_{m\to\infty} \left\| \frac{\partial}{\partial \nu} v^{(m)} \right\|_{L^2(\Sigma)} = 0$$

and so $\{v^{(m)}\}$ converges to zero uniformly in every closed set interior to Ω (see [4]). This implies that $v(y)=0$, $y \in \Omega$. In order to study $v(y)$ in $\mathbf{R}^n - \bar{\Omega}$, we suppose $0 \in \Omega$ and we introduce the transformation $\tilde{y} = y/|y|^2$. Therefore $\mathbf{R}^n - \bar{\Omega}$ is mapped onto a bounded domain $\tilde{\Omega}$. Moreover the function $\tilde{v}(\tilde{y}) = |\tilde{y}|^{n-4}\, v(\tilde{y}/|\tilde{y}|^2)$ is biharmonic in $\tilde{\Omega} - \{0\}$ and it is bounded with its first and second derivatives in a neighborhood of $\{0\}$ (this easily follows from the behaviour of $v(y)$ at infinity) and then it is biharmonic in $\tilde{\Omega}$ (see [13]). Arguing as above we can prove that $\tilde{v}(\tilde{y}) = 0$ in $\tilde{\Omega}$ and so $v(y) \equiv 0$, *i.e.* $(\bar{\zeta}_1, \ldots, \bar{\zeta}_n)$ is of the first kind. This is a contradiction. In the same way it is possible to prove that two linearly independent eigensolutions of the second kind do not exist. If there exists an eigensolution of the second kind, there exists a point $\bar{x} \in \mathbf{R}^n - \bar{\Omega}$ such that

$$\int_{\Sigma} \bar{\zeta}_h(x) \wedge d_x \left[\frac{\partial}{\partial x_h} F(x,\bar{x}) \right] \neq 0.$$

Let us fix $(\bar{\zeta}_1, \ldots, \bar{\zeta}_n)$ and $\bar{x}$ in such a way that

$$\int_{\Sigma} \bar{\zeta}_h(x) \wedge d_x \left[\frac{\partial}{\partial x_h} F(x,\bar{x}) \right] = 1$$

If eigensolutions of the second kind do not exist we set $(\bar{\zeta}_1, \ldots, \bar{\zeta}_n) = (0, \ldots, 0)$.

6. The existence, uniqueness and representation theorem

We are now in a position to prove the existence, uniqueness and representation theorem for the following problem

$$(6.1)\qquad \begin{cases} u \in C^{2+\lambda}(\bar{\Omega}) \cap C^{\omega}(\Omega) & \Delta_4 u = 0 \ \text{ in } \Omega, \\ u|_{\Sigma} = g_0, \quad (\partial u/\partial x_h)|_{\Sigma} = g_h & (h = 1, \dots, n) \quad (\Delta_4 = \Delta_2 \Delta_2). \end{cases}$$

VII. *Let* $g_0, g_h \in C^{1+\lambda}(\Sigma)$. *There exists the solution of* (6.1) *if and only if*

$$(6.2)\qquad dg_0 = g_h dx^h \qquad (on \quad \Sigma).$$

The solution is unique. Moreover there exists $(\varphi_1, \dots, \varphi_n) \in [C^{\lambda}(\Sigma)]^n$ *and* $a_0, a_h, b \in \mathbf{R}$ *such that the following representation holds*:

$$(6.3)\qquad u(x) = \int_{\Sigma} \varphi_h(y) \frac{\partial}{\partial y_h} F(x,y)\, d\sigma_y + b F(x, \bar{x}) + a_h x_h + a_0, \qquad x \in \Omega.$$

The necessity of condition (6.2) is obvious. Conversely, set $f_h(x) = dg_h(x)$, $\bar{f}_h(x) = f_h(x) - b d_x [\partial F(x, \bar{x})/\partial x_h]$, where

$$b = \int_{\Sigma} \bar{\zeta}_h \wedge f_h .$$

$(\bar{f}_1, \dots, \bar{f}_n)$ satisfies all the compatibility conditions (4.6) and then (Theorem VI) there exists a biharmonic simple layer potential $w(x)$ such that $d(\partial w/\partial x_h) = \bar{f}_h \, (h = 1, \dots, n)$. Using (6.2) it follows that the solution of problem (6.1) is given by (6.3), where $(\varphi_1, \dots, \varphi_n) \in [L^2(\Sigma)]^n$. On the other hand we can represent the function

$$w(x) = \int_{\Sigma} \varphi_h(y) \frac{\partial}{\partial y_h} F(x,y)\, d\sigma_y$$

by means of (3.1), (3.5). Using this representation and the reduction obtained in section 3, it is possible to show that we can take $(\varphi_1, \dots, \varphi_n) \in [C^{\lambda}(\Sigma)]^n$. This implies that $u(x) \in C^{2+\lambda}(\bar{\Omega})$. The uniqueness of the solution can be proved by standard arguments.

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