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Prime divisors of conjugacy class lengths in finite groups


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Teoria dei gruppi. — Prime divisors of conjugacy class lengths in finite groups. Nota di CARLO CASOLO, presentata (*) dal Socio G. ZACHER.

ABSTRACT. — We show that in a finite group $G$ which is $p$-nilpotent for at most one prime dividing its order, there exists an element whose conjugacy class length is divisible by more than half of the primes dividing $|G/Z(G)|$.

KEY WORDS: Finite groups; Conjugacy classes; Lengths.

INTRODUCTION

If $G$ is a finite group and $g \in G$, we denote by $\sigma_G(g)$ the set of all prime divisors of $|G: C_G(g)|$, the length of the conjugacy class of $g$. Also we put:

$$p'(G) = \bigcup_{g \in G} \sigma_G(g) \quad \text{and} \quad \sigma'(G) = \max_{g \in G} |\sigma_G(g)|.$$

It is easy to show that $p'(G)$ is the set of all prime divisors of $|G/Z(G)|$. In this Note we prove (Corollary 2), by an elementary method, that if $G$ is a finite group which is not $p$-nilpotent for two or more primes dividing its order, then

$$|p'(G)| \leq 2\sigma'(G).$$

At the International Group Theory Conference in Bressanone 1989, Prof. Huppert asked whether such a bound holds for every finite soluble group. The motivation for this question comes from the observation of a not yet well understood parallelism between results on characters and results on conjugacy classes.

There is for example the conjecture that for every finite soluble group $G$, $|\varphi(G)| \leq 2\sigma(G)$, where $\varphi(G)$ is the set of all primes dividing the degree of some irreducible complex character of $G$, and $\sigma(G)$ is the maximum number of distinct primes dividing the degree of a single irreducible character of $G$. The best result published so far in this direction, due to D. Gluck and O. Manz [3], states that for every finite soluble group $G$, $|\varphi(G)| \leq 3\sigma(G) + 32$.

Our result, which gives an affirmative answer to Huppert's question on conjugacy classes for a large class of finite groups, seems also to indicate that restricting to soluble groups might not be unavoidable in this contest: an immediate corollary of our Theorem is that the bound $(\ast)$ holds in every finite perfect group $G$.

We recall that it follows from results of D. Chillag and M. Herzog [1], that $|\varphi'(G)| \leq 2$ for all finite groups $G$ with $\sigma'(G) = 1$ (such a group is necessarily soluble); also P. Ferguson [2] has shown that $|\varphi'(G)| \leq 4$ for every finite soluble group $G$ with

(*) Nella seduta del 15 dicembre 1990.
\( \sigma'(G) = 2 \). In a subsequent work we will show that the inequality (*) holds for every finite group \( G \) such that \( \sigma'(G) = 2 \), and every finite soluble group with \( \sigma'(G) = 3 \).

All groups considered in this Note are finite.

PROOFS. We start by fixing some more notations. If \( G \) is a group, we denote by \( \pi(G) \) the set of all prime divisors of \( |G| \), and by \( \Delta(G) \) the set of all primes \( p \in \pi(G) \) such that \( G \) is not \( p \)-nilpotent with abelian Sylow \( p \)-subgroups. If \( p \in \pi(G) \) we let \( G_p \) be a Sylow \( p \)-subgroup of \( G \) and put \( n_p(G) = |N_G(G_p) : C_G(G_p)| \) (we simply write \( n_p \) when it will be obvious to which group we refer). Now, \( n_p(G) = 1 \) if and only if \( G_p \trianglelefteq Z(N_G(G_p)) \); thus Burnside's criterion for \( p \)-nilpotency implies that \( n_p(G) = 1 \) if and only if \( G \) is \( p \)-nilpotent with abelian Sylow \( p \)-subgroups.

In particular \( \Delta(G) = \{ p \in \pi(G); n_p(G) \neq 1 \} \).

**Theorem.** Let \( G \) be a non-abelian group. Then:

\[ \sigma'(G) > \sum_{p \in \Delta(G)} \frac{n_p - 1}{n_p} \]

**Proof.** For every \( p \in \pi(G) \), we put \( \mathcal{L}_p = \{ g \in G; p \in \sigma_G(g) \} = \{ g \in G; p \) divides \( |G : C_G(g)| \} \).

Let \( x \in G \); then \( x \notin \mathcal{L}_p \) if and only if \( p \) does not divide \( |G : C_G(x)| \), if and only if there exists a Sylow \( p \)-subgroup \( G_p \) of \( G \) such that \( G_p \trianglelefteq C_G(x) \). Thus:

\[ G \setminus \mathcal{L}_p = \bigcup_{g \in G} C_G(G_p^g). \]

Hence

\[ |G| - |\mathcal{L}_p| \leq |G|(|C_G(G_p)| - 1)/|N_G(G_p)| + 1 \leq |G|/|N_G(G_p)| : C_G(G_p)| = |G|/n_p \]

and so:

\[ (1) \quad |\mathcal{L}_p| \geq |G| - \frac{|G|}{n_p} = |G| \left( \frac{n_p - 1}{n_p} \right). \]

In particular, we have that if \( p \in \Delta(G) \), then \( |\mathcal{L}_p| \geq |G|/2 \). We now consider the following subset of \( \pi(G) \times G \): \( S = \{ (p, x) \in \pi(G) \times G; p \in \sigma_G(x) \} \).

Observing that \( (p, x) \in S \) if and only if \( x \in \mathcal{L}_p \), and by counting the number of elements of \( S \) in two ways, we get:

\[ \sum_{p \in \pi(G)} |\mathcal{L}_p| = |S| = \sum_{x \in G} |\sigma_G(x)|. \]

Since \( \sigma_G(1) = \emptyset \), we may write:

\[ (2) \quad \sum_{x \in G^*} |\sigma_G(x)| = \sum_{p \in \pi(G)} |\mathcal{L}_p| \]

Hence, by formula (1):

\[ \sigma'(G)(|G| - 1) \geq \sum_{p \in \pi(G)} \left( |G| \frac{n_p - 1}{n_p} \right). \]
Now, for $p \in \pi(G) \setminus \Delta(G)$, it is $n_p - 1 = 0$, so we have:

$$|G| \sigma'(G) > |G| \sum_{p \in \Delta(G)} \frac{n_p - 1}{n_p},$$

and the desired result.

**Corollary 1.** Let $G$ be a non-abelian group. Then: $|\Delta(G)| < 2\sigma'(G)$.

**Proof.** By the Theorem:

$$\sigma'(G) > \sum_{p \in \Delta(G)} \frac{n_p - 1}{n_p} \geq |\Delta(G)| \cdot \frac{1}{2}.$$ 

**Corollary 2.** Let $G$ be a finite group and assume that the quotient group of $G$ by the derived subgroup contains a subgroup isomorphic to a Sylow $p$-subgroup of $G$ for at most one prime $p \in \pi(G)$, then: $|\rho'(G)| \leq 2\sigma'(G)$.

**Proof.** The hypothesis on $G$ implies $|\Delta(G)| \leq |\rho'(G)| - 1$ hence, by Corollary 1: $2\sigma'(G) > |\rho'(G)| - 1$ and so $2\sigma'(G) \geq |\rho'(G)|$.

**References**

