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Holomorphic isometries of Cartan domains of type one

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Geometria. — *Holomorphic isometries of Cartan domains of type one.* Nota (*) del Socio Edoardo Vesentini.

Abstract. — Holomorphic isometries for the Kobayashi metric of a class of Cartan domains are characterized.

Key words: Cartan domain; Kobayashi metric; Holomorphic isometry.

Riassunto. — *Isometrie olomorfe di domini di Cartan di tipo uno.* Si caratterizzano le isometrie olomorfe per la metrica di Kobayashi di una classe di domini di Cartan.

Let $\mathcal{H}$ and $\mathcal{K}$ be two complex Hilbert spaces and let B be the open unit ball of the complex Banach space $\mathcal{L}(\mathcal{K}, \mathcal{H})$ of all bounded linear mappings from $\mathcal{K}$ to $\mathcal{H}$. Extending to infinite dimensions a classical terminology, B has been given the name of a Cartan domain of type one. This domain is homogeneous, *i.e.*, the group $\mathrm{Aut}\,B$ of all holomorphic automorphisms of B acts transitively. Since B is an open, bounded, circular neighborhood of 0, a theorem by H. Cartan [2] implies that the stability group $(\mathrm{Aut}\,B)_0$ of 0 in $\mathrm{Aut}\,B$ is linear, or, more exactly, every element of $(\mathrm{Aut}\,B)_0$ is the restriction to B of a linear isometric isomorphism of $\mathcal{L}(\mathcal{K}, \mathcal{H})$. This fact, coupled with the explicit knowledge of a transitive subgroup of $\mathrm{Aut}\,B$, leads to a complete description of the latter group. This description was carried out by H. Klingen [4, 5] when both $\mathcal{H}$ and $\mathcal{K}$ have finite dimension, and by T. Franzoni [1] in the general case. The elements of $\mathrm{Aut}\,B$ turn out to be invertible rational functions which are the operator-valued analogues of the Moebius transformations acting on the unit disc of $\mathbf{C}$.

Let $\mathrm{Iso}\,B$ be the semigroup of all holomorphic maps of B into B which are isometries for the (Carathéodory-) Kobayashi metric of B [2]. Since this metric is invariant under all holomorphic automorphisms, then $\mathrm{Aut}\,B$ is a subgroup (actually the maximum subgroup) of $\mathrm{Iso}\,B$. It coincides with $\mathrm{Iso}\,B$ when both $\mathcal{H}$ and $\mathcal{K}$ have finite dimension, and is properly contained in $\mathrm{Iso}\,B$ otherwise. Thus, if at least one of the two spaces $\mathcal{H}$ and $\mathcal{K}$ has infinite dimension, the question naturally arises to describe $\mathrm{Iso}\,B$. An example constructed in [7] in the case in which B is the open unit ball of the C^*-algebra $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$ ($\dim_{\mathbf{C}} \mathcal{H} = \infty$) exhibits a non-linear element of $\mathrm{Iso}\,B$ fixing 0, showing thereby that H. Cartan's theorem fails for $\mathrm{Iso}\,B$ and leaving completely open the characterization of this semigroup in the infinite dimensional case.

The main purpose of this *Note* is to show that H. Cartan's theorem holds for $\mathrm{Iso}\,B$ when one of the two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ has finite dimension, and to characterize the stability semigroup $(\mathrm{Iso}\,B)_0$ of 0 in $\mathrm{Iso}\,B$ within the semigroup of all linear operators acting on $\mathcal{L}(\mathcal{K}, \mathcal{H})$. This characterization yields a description

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of Iso$B$ in terms of non-invertible, operator-valued «Moebius transformations» which have been investigated in [8].

1. A J^*-algebra [3] is a closed linear subspace $\mathcal{E}$ of $\mathcal{L}(\mathcal{K}, \mathcal{H})$ such that, if $X \in \mathcal{E}$, $XX^*X \in \mathcal{E}$. Here X^* denotes the adjoint operator of X. The space $\mathcal{L}(\mathcal{K}, \mathcal{H})$ itself is a J^*-algebra. If $\mathcal{E}$ and $\mathcal{F}$ are J^*-algebras, a continuous linear map $L\colon \mathcal{E} \to \mathcal{F}$ is called a J^*-homomorphism if

$$L(XX^*X) = L(X)\, L(X)^*\, L(X) \tag{1.1}$$

for all $X \in \mathcal{E}$. A simple polarization argument yields then

$$L(XY^*X) = L(X)\, L(Y)^*\, L(X) \tag{1.2}$$

for all X, Y in $\mathcal{E}$. Since $XY^*Z + ZY^*X = (X+Z)\, Y^*(X+Z) - XY^*X - ZY^*Z$, (1.2) yields

$$L(XY^*Z + ZY^*X) = L(X)\, L(Y)^*\, L(Z) + L(Z)\, L(Y)^*\, L(X) \tag{1.3}$$

for all X, Y, Z in $\mathcal{E}$.

The unit ball B of $\mathcal{E}$ is a bounded homogeneous domain.

In [3] L. A. Harris proved that every J^*-homomorphism of $\mathcal{E}$ into $\mathcal{F}$ is a linear isometry, and furthermore [3, Theorem 4] that if $L\colon \mathcal{E} \to \mathcal{F}$ is a linear surjective isometry, then L is a J^*-homomorphism. Actually, a direct inspection of Harris' argument shows that he proved slightly more, namely the following. Let $B_{\mathcal{E}}$ and $B_{\mathcal{F}}$ be the open unit balls of $\mathcal{E}$ and $\mathcal{F}$, and let Iso $(B_{\mathcal{E}}, B_{\mathcal{F}})$ be the set of all holomorphic maps of $B_{\mathcal{E}}$ into $B_{\mathcal{F}}$ which are isometries for the respective Kobayashi metrics. The following proposition holds.

PROPOSITION 1. *If every $L \in$ Iso $(B_{\mathcal{E}}, B_{\mathcal{F}})$ such that $L(0) = 0$, is the restriction to $B_{\mathcal{E}}$ of a linear mapping of $\mathcal{E}$ into $\mathcal{F}$, then every such L is the restriction to $B_{\mathcal{E}}$ of a J^*-homomorphism.*

2. The closed subspace $\mathcal{L}_0(\mathcal{K}, \mathcal{H}) \subset \mathcal{L}(\mathcal{K}, \mathcal{H})$ of all compact operators from $\mathcal{K}$ to $\mathcal{H}$ is a J^*-algebra. Since $\mathcal{L}_0(\mathcal{K}, \mathcal{H})$ and $\mathcal{L}_0(\mathcal{H}, \mathcal{K})$ are J^*-isomorphic, it will be assumed henceforth that $\dim_{\mathbf{C}} \mathcal{K} \leq \dim_{\mathbf{C}} \mathcal{H}$. Every $X \in \mathcal{L}_0(\mathcal{K}, \mathcal{H})$ is expressed by

$$X = \sum \alpha_\nu f_\nu \otimes e_\nu^* \tag{2.1}$$

where: $\alpha_1 \geq \alpha_2 \geq \ldots > 0$ are the singular values of X, *i.e.*, $\alpha_1^2, \alpha_2^2, \ldots$ are the non-vanishing eigenvalues of X^*X counted with their (finite) multiplicities; e_ν is an eigenvector of X^*X corresponding to the eigenvalue α_ν^2: $\{e_1, e_2, \ldots\}$ is an orthonormal system in $\mathcal{K}$; $f_\nu = \alpha_\nu^{-1} X e_\nu$, and $(f_\nu | f_\mu)_{\mathcal{H}} = \delta_{\nu\mu}$; $(f_\nu \otimes e_\nu^*)(x) = (x|e_\nu)_{\mathcal{K}} f_\nu$ for all $x \in \mathcal{K}$. The operator X is a partial isometry if, and only if, X^*X is an orthogonal projector. Since

$$X^*X = \sum \alpha_\nu^2\, e_\nu \otimes e_\nu^*, \tag{2.2}$$

and $(X^*X)^2 = \sum \alpha_\nu^4\, e_\nu \otimes e_\nu^*$, that happens if, and only if, $\alpha_\nu = 1$ for all ν. As a consequence, the set of all α_ν appearing in (2.1) is finite. Denoting by N its cardinality, every partial isometry in $\mathcal{L}_0(\mathcal{K}, \mathcal{H})$ is given by

$$X = \sum_1^N f_\nu \otimes e_\nu^* .$$

Let X be the partial isometry in $\mathcal{L}_0(\mathcal{K}, \mathcal{H})$ represented by this latter formula. Denoting by $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$ the identity operators in $\mathcal{H}$ and in $\mathcal{K}$, and by $\mathcal{K}_0$ and $\mathcal{H}_0$ the closed subspaces of $\mathcal{K}$ and $\mathcal{H}$ spanned by $\{e_1, \dots, e_N\}$ and by $\{f_1, \dots, f_N\}$, $I_{\mathcal{K}} - X^*X$ and $I_{\mathcal{H}} - XX^*$ are the orthogonal projectors onto $\mathcal{K}_0^\perp$ and onto $\mathcal{H}_0^\perp$. If, and only if, either $\mathcal{H} = \mathcal{H}_0$ or $\mathcal{K} = \mathcal{K}_0$, then $(I_{\mathcal{H}} - XX^*)Y(I_{\mathcal{K}} - X^*X) = 0$ for all $Y \in \mathcal{L}_0(\mathcal{K}, \mathcal{H})$. By the Kadison-Harris theorem [3, Theorem 11] that proves

PROPOSITION 2. *If both $\mathcal{H}$ and $\mathcal{K}$ have infinite dimension, the closed unit ball $\overline{B}$ of $\mathcal{L}_0(\mathcal{K}, \mathcal{H})$ has no extreme points. If $\mathcal{K}$ has finite dimension, and* $\dim_C \mathcal{K} \leqslant \dim_C \mathcal{H}$, *the extreme points of $\overline{B}$ are all the linear isometries of $\mathcal{K}$ into $\mathcal{H}$.*

3. Let $\dim_C \mathcal{K} = n < \infty$, $\dim_C \mathcal{H} = \infty$. Every $X \in \mathcal{L}_0(\mathcal{K}, \mathcal{H}) \setminus \{0\}$ is expressed by (2.1) where the summation runs over all $\nu = 1, \dots, N$, with $1 \leqslant N \leqslant n$. Then

$$\det(I_{\mathcal{K}} - X^*X) = \prod_1^N (1 - \alpha_\nu^2).$$

Since $\|X\| = \max\{\alpha_\nu : \nu = 1, \dots, N\}$, then $X \in \partial B$ if, and only if, $\det(I_{\mathcal{K}} - X^*X) = 0$.

Let $F \in (\mathrm{Iso}\, B)_0$ and let $L = dF(0) \in \mathcal{L}(\mathcal{L}(\mathcal{K}, \mathcal{H}))$.

Denoting by $\varkappa : B \times \mathcal{L}(\mathcal{K}, \mathcal{H}) \to \boldsymbol{R}_+$ the Kobayashi infinitesimal metric on B [2], then $\varkappa(0; L(X)) = \varkappa(0; X)$ for all $X \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. Since $\varkappa(0; \cdot) = \|\cdot\|$, then $\|L(X)\| = \|X\|$ for all $X \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ *i.e.*, L is a linear isometry of $\mathcal{L}(\mathcal{K}, \mathcal{H})$ into itself.

It will be shown now that $F(X) = L(X)$ for all $X \in B$. This result will be established by using an argument first devised by C. L. Siegel in [6] in the case of the Siegel disc in $\boldsymbol{C}^{n(n+1)/2}$. For X given by (2.1),

$$L(X) = \sum_1^N \alpha_\nu L(f_\nu \otimes e_\nu^*).$$

The set of all $X \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ such that $N = n$ and $\alpha_1, \dots, \alpha_n$ are distinct is a non-empty dense open set $S \subset \mathcal{L}(\mathcal{K}, \mathcal{H})$. For all X, $\det(I_{\mathcal{K}} - L(X)^* L(X))$ is a polynomial of degree $2n$ in $\alpha_1, \dots, \alpha_n$, whose constant term equals 1. As before $L(X) \in \partial B$ if, and only if, $\det(I_{\mathcal{K}} - L(X)^* L(X)) = 0$. On the other hand, since L is an isometry, $L(X) \in \partial B$ if, and only if,

$$\prod_1^N (1 - \alpha_\nu^2) = 0.$$

Hence

$$\prod_1^N (1 - \alpha_\nu^2)$$

divides $\det(I_{\mathcal{K}} - L(X)^* L(X))$ for all $X \in S$, and in conclusion

$$\det(I_{\mathcal{K}} - L(X)^* L(X)) = \det(I_{\mathcal{K}} - X^* X)$$

for all $X \in S$ and therefore for all $X \in \mathcal{L}(\mathcal{K}, \mathcal{H})$.

Thus

$$L(X) = \sum_{\nu=1}^{N} \alpha_\nu (f'_\nu \otimes e'^*_\nu) \tag{3.1}$$

for suitable choices of the orthonormal systems $\{e'_1, \ldots, e'_N\}$, and $\{f'_1, \ldots, f'_N\}$ in $\mathcal{K}$ and $\mathcal{H}$ respectively. If, in particular, X is a linear isometry, then also $L(X)$ is a linear isometry. Thus, by Proposition 2, L maps the set of all extreme points of $\overline{B}$ into itself. By the Schwarz lemma [3, Theorem 10], $F = L_{|B}$. Proposition 1 yelds then

PROPOSITION 3. *Let* $\dim_C \mathcal{K} = n < \infty$, $\dim_C \mathcal{H} = \infty$. *If* $F \in \operatorname{Iso} B$ *fixes* 0, *there exists a* J^**-homomorphism* L *of* $\mathcal{L}(\mathcal{K}, \mathcal{H})$ *into itself whose restriction to* B *is* F.

4. Let L be a J^*-homomorphism of $\mathcal{L}_0(\mathcal{K}, \mathcal{H})$ into itself. If $e \in \mathcal{K}$, $f \in \mathcal{H}$ are such that $\|e\| = \|f\| = 1$, and if $X = f \otimes e^*$, then $XX^*X = X$. Setting $Y = L(X)$, then by (1.1) $YY^*Y = L(XX^*X) = L(X) = Y$, whence $Y^*YY^*Y = Y^*Y$, $YY^*YY^* = YY^*$, *i.e.*, Y^*Y and YY^* are orthogonal projectors in $\mathcal{K}$ and in $\mathcal{H}$.

If $e_1 \in \mathcal{K}, f_1 \in \mathcal{H}$ are such that $\|e_1\| = \|f_1\| = 1$, $e \perp e_1, f \perp f_1$, and if $X_1 = f_1 \otimes e_1^*$, $Y_1 = L(X_1)$, then $X^*X_1 = (\cdot|e_1)_{\mathcal{K}} X^* f_1 = (\cdot|e_1)_{\mathcal{K}} (f_1|f)_{\mathcal{H}} e = 0$, $X_1 X^* = (\cdot|f)_{\mathcal{K}} X_1 e = (\cdot|f)_{\mathcal{H}} (e|e_1)_{\mathcal{K}} f_1 = 0$, so that, by (1.3), $YY^*Y_1 + Y_1Y^*Y = L(XX^*X_1 + X_1X^*X) = 0$, which is readily seen [3] to be equivalent to $Y_1Y^* = 0$, $Y^*Y_1 = 0$. That proves

LEMMA 4. *The orthogonal projectors* Y^*Y *and* $Y_1^*Y_1$ *in* $\mathcal{K}$ *are orthogonal to each other. Similarly, the orthogonal projectors* YY^* *and* $Y_1Y_1^*$ *in* $\mathcal{H}$ *are orthogonal to each other.*

It will be assumed henceforth that $n = \dim_C \mathcal{K} < \infty$, $\dim_C \mathcal{H} = \infty$. For X given by (2.1) with $\nu = 1, \ldots, N \leq n$, $Y = L(X)$ is expressed by (3.1) and therefore

$$Y^*Y = \sum_{\nu=1}^{N} \alpha_\nu^2 (e'_\nu \otimes e'^*_\nu). \tag{4.1}$$

If V' is a unitary operator in $\mathcal{K}$ such that $V' e_\nu = e'_\nu$ for $\nu = 1, \ldots, N$, (2.2) and (4.1) yield

$$Y^*Y = \sum_{\nu=1}^{N} \alpha_\nu^2 V' e_\nu \otimes (V'^* e_\nu)^* = V'^* X^* X V'.$$

If U' is a linear isometry of $\mathcal{H}$ such that $U' f_\nu = f'_\nu$ for $\nu = 1, \ldots, N$, then

$$Y = \sum_{\nu=1}^{N} \alpha_\nu U' f_\nu \otimes (V'^* e_\nu)^* = U' X V'.$$

Note that the choices of V' and U' depend only on $\{e_1, \ldots, e_N\}$, $\{e'_1, \ldots, e'_N\}$, and $\{f_1, \ldots, f_N\}$, $\{f'_1, \ldots, f'_N\}$, respectively. Fix now an orthonormal base $\{e_1, \ldots, e_n\}$ in $\mathcal{K}$ and an orthonormal base $\{f_\mu : \mu \in M\}$ in $\mathcal{H}$. For $\nu = 1, \ldots, n$, and $\mu \in M$, let $X_{\mu\nu} = f_\mu \otimes e_\nu^*$, $Y_{\mu\nu} = L(X_{\mu\nu})$.

There exist a unitary operator V_ν in $\mathcal{K}$ and a linear isometry U_μ in $\mathcal{H}$, depending only on ν and on μ respectively, such that $Y_{\mu\nu} = U_\mu X_{\mu\nu} V_\nu = U_\mu f_\mu \otimes (V_\nu^* e_\nu)^*$.

Hence the orthogonal projector $Y_{\mu\nu}^* Y_{\mu\nu} = V_\nu^* e_\nu \otimes (V_\nu^* e_\nu)^*$ maps $\mathcal{K}$ onto the complex line generated by $V_\nu^* e_\nu$ and does not depend on μ: $Y_{\mu\nu}^* Y_{\mu\nu} = Y_{\mu'\nu}^* Y_{\mu'\nu}$ for all μ, μ' in M. Setting $P_\nu = Y_{\mu\nu}^* Y_{\mu\nu}$, Lemma 4 implies that the orthogonal projectors P_ν and $P_{\nu'}$ are orthogonal to each other, *i.e.*, that $V_\nu^* e_\nu$ is orthogonal to $V_{\nu'}^* e_{\nu'}$ whenever $\nu \neq \nu'$. A similar argument shows that $Y_{\mu\nu} Y_{\mu\nu}^* = Y_{\mu\nu'} Y_{\mu\nu'}^*$ for all ν, $\nu' = 1, \dots, n$. Hence the orthogonal projector $Q_\mu = Y_{\mu\nu} Y_{\mu\nu}^* = U_\mu f_\mu \otimes (U_\mu f_\mu)^*$ maps $\mathcal{H}$ onto the complex line generated by $U_\mu f_\mu$, and Q_μ and $Q_{\mu'}$ are orthogonal to each other, *i.e.*, $U_\mu f_\mu$ is orthogonal to $U_{\mu'} f_{\mu'}$ whenever $f_\mu \neq f_{\mu'}$. In conclusion, there exists an orthonormal base $\{e_1', \dots, e_n'\}$ in $\mathcal{K}$ and orthonormal system $\{f_\mu'\}_{\mu \in M}$ in $\mathcal{H}$ such that

$$Y_{\mu\nu} = f_\mu' \otimes e_\nu'^* . \tag{4.2}$$

If V is a unitary operator in $\mathcal{K}$ such that $V e_\nu = e_\nu'$ for $\nu = 1, \dots, n$ and if U is a linear isometry in $\mathcal{H}$ such that $U f_\mu = f_\mu'$ for all $\mu \in M$, (4.2) yields $L(X_{\mu\nu}) = U X_{\mu\nu} V$ $(\nu = 1, \dots, n;\ \mu \in M)$.

5. Every $Z \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ is expressed by

$$Z = \sum_{\lambda=1}^{N} \beta_\lambda l_\lambda \otimes g_\lambda^*$$

where: $0 \leq N \leq n$; $\beta_1 \geq \dots \geq \beta_N > 0$ are the singular values of Z; $l_1, \dots, l_N$ and $g_1, \dots, g_N$ are suitably chosen orthonormal systems in $\mathcal{H}$ and $\mathcal{K}$ respectively. Since

$$g_\lambda = \sum_{\nu=1}^{n} (g_\lambda | e_\nu)_{\mathcal{K}} e_\nu , \qquad l_\lambda = \sum_{\mu \in M} (l_\lambda | f_\mu)_{\mathcal{H}} f_\mu ,$$

then

$$Z = \sum_{\nu=1}^{n} \sum_{\mu \in M} \left(\sum_{\lambda=1}^{N} \beta_\lambda (e_\nu | g_\lambda)_{\mathcal{K}} (l_\lambda | f_\mu)_{\mathcal{H}} \right) X_{\mu\nu} ,$$

whence

$$Z = \sum_{\nu=1}^{n} \sum_{\mu \in M} (Z e_\nu | f_\mu)_{\mathcal{H}} X_{\mu\nu} . \tag{5.1}$$

LEMMA 5. *The right hand side of (5.1) converges to Z in the Banach space $\mathcal{L}(\mathcal{K}, \mathcal{H})$.*

PROOF. Since

$$\|Z e_\nu\|^2 = \sum_{\mu \in M} |(Z e_\nu | f_\mu)_{\mathcal{H}}|^2 ,$$

there exists a (finite or) countable set $M_0 \subset M$ such that $(Z e_\nu | f_\mu)_{\mathcal{H}} = 0$ whenever $\mu \notin M_0$ and $\nu = 1, \dots, n$. For any $\varepsilon > 0$ there is a finite set $M_1 \subset M_0$ such that

$$\sum_{\mu \notin M_1} |(Z e_\nu | f_\nu)_{\mathcal{H}}|^2 < \varepsilon^2 . \tag{5.2}$$

Let

$$K = \left\| \sum_{\mu \notin M_1} \sum_{\nu=1}^{n} (Ze_\nu \mid f_\mu)_{\mathcal{H}} X_{\mu\nu} \right\|.$$

Then

$$K^2 = \sup\left\{ \left\| \sum_{\mu \notin M_1} \sum_{\nu=1}^{n} (Ze_\nu \mid f_\mu)_{\mathcal{H}} X_{\mu\nu}(\xi) \right\|_{\mathcal{H}}^2 : \|\xi\|_{\mathcal{K}} \leq 1 \right\} =$$

$$= \sup\left\{ \left(\sum_{\mu \notin M_1} \sum_{\nu=1}^{n} (Ze_\nu \mid f_\mu)_{\mathcal{H}} X_{\mu\nu}(\xi) \;\middle|\; \sum_{\mu' \notin M_1} \sum_{\nu'=1}^{n} (Ze_{\nu'} \mid f_{\mu'})_{\mathcal{H}} X_{\mu'\nu'}(\xi) \right)_{\mathcal{H}} : \|\xi\|_{\mathcal{K}} \leq 1 \right\} =$$

$$= \sup\left\{ \sum_{\mu \notin M_1} \sum_{\nu,\nu'=1}^{n} ((Ze_\nu \mid f_\mu)_{\mathcal{H}} X_{\mu\nu}(\xi) | (Ze_{\nu'} \mid f_{\mu'})_{\mathcal{H}} X_{\mu\nu'}(\xi))_{\mathcal{H}} : \|\xi\|_{\mathcal{K}} \leq 1 \right\} \leq$$

$$\leq \frac{n}{2} \sup\left\{ \sum_{\mu \notin M_1} \left[\sum_{\nu=1}^{n} |(Ze_\nu \mid f_\mu)_{\mathcal{H}}|^2 \, \|X_{\mu\nu}(\xi)\|_{\mathcal{H}}^2 + \sum_{\nu=1}^{n} |(Ze_\nu \mid f_\mu)_{\mathcal{H}}|^2 \, \|X_{\mu\nu}(\xi)\|_{\mathcal{H}}^2 \right] : \|\xi\|_{\mathcal{K}} \leq 1 \right\} =$$

$$= n \sup\left\{ \sum_{\mu \notin M_1} \sum_{\nu=1}^{n} |(Ze_\nu \mid f_\mu)_{\mathcal{H}}|^2 \, \|X_{\mu\nu}(\xi)\|_{\mathcal{H}}^2 : \|\xi\|_{\mathcal{K}} \leq 1 \right\} = n \sum_{\mu \notin M_1} \sum_{\nu=1}^{n} |(Ze_\nu \mid f_\mu)_{\mathcal{H}}|^2 \, \|X_{\mu\nu}\|^2 =$$

$$= n \sum_{\mu \notin M_1} \sum_{\nu=1}^{n} |(Ze_\nu \mid f_\mu)_{\mathcal{H}}|^2,$$

and (5.2) yields $K < \sqrt{n}\varepsilon$. Q.E.D.

As a consequence

$$L(Z) = \sum_{\mu \in M} \sum_{\nu=1}^{n} (Ze_\nu \mid f_\mu)_{\mathcal{H}} L(X_{\mu\nu}) = \sum_{\mu \in M} \sum_{\nu=1}^{n} (Ze_\nu \mid f_\mu)_{\mathcal{H}} U X_{\mu\nu} V =$$

$$= U\left(\sum_{\mu \in M} \sum_{\nu=1}^{n} (Ze_\nu \mid f_\mu)_{\mathcal{H}} X_{\mu\nu} \right) V = UZV,$$

proving thereby

THEOREM I. *If* $\dim_{\mathbb{C}} \mathcal{K} < \infty$, *for any J^*-homomorphism L of $\mathcal{L}(\mathcal{K}, \mathcal{H})$ into itself, there exist a unitary operator V in $\mathcal{K}$ and a linear isometry U in $\mathcal{H}$ such that $L(Z) = UZV$ for all $Z \in \mathcal{L}(\mathcal{K}, \mathcal{H})$.*

COROLLARY. *If* $\dim_{\mathbb{C}} \mathcal{K} < \infty$, *for any $F \in (\mathrm{Iso}\, B)_0$ there are a unitary operator V in $\mathcal{K}$ and a linear isometry U in $\mathcal{H}$ such that F is the restriction to B of the linear map $Z \to UZV$ $(Z \in \mathcal{L}(\mathcal{K}, \mathcal{H}))$.*

6. Let J be the operator on the Hilbert space direct sum $\mathcal{H} \oplus \mathcal{K}$ defined by the matrix

$$J = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & -I_{\mathcal{K}} \end{pmatrix},$$

and let Λ be the semigroup of all $A \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ such that

$$A^* J A = J. \tag{6.1}$$

Let Γ be the maximum subgroup of Λ consisting of all $A \in \Lambda$ which are continuously invertible in $\mathcal{L}(\mathcal{H} \oplus \mathcal{K})$. Any $A \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ is represented by a matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{11} \in \mathcal{L}(\mathcal{H})$, $A_{22} \in \mathcal{L}(\mathcal{K})$, $A_{12} \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $A_{21} \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. Condition (6.1) is equivalent to

$$A_{11}^* A_{11} - A_{21}^* A_{21} = I_{\mathcal{H}}, \tag{6.2}$$

$$A_{22}^* A_{22} - A_{12}^* A_{12} = I_{\mathcal{K}}, \tag{6.3}$$

$$A_{12}^* A_{11} - A_{22}^* A_{21} = 0. \tag{6.4}$$

It has been shown in [8] that: if $\dim_{\mathbf{C}} \mathcal{K} < \infty$: $A_{21} Z + A_{22}$ is continuously invertible in $\mathcal{L}(\mathcal{K})$ for any $Z \in B$; the holomorphic function $\widetilde{A}: B \to \mathcal{L}(\mathcal{K}, \mathcal{H})$ defined by

$$\widetilde{A}(Z) = (A_{11} Z + A_{12})(A_{21} Z + A_{22})^{-1} \tag{6.5}$$

maps B into B and is an element of $\operatorname{Iso} B$; the function $A \to \widetilde{A}$ defines a homomorphism of Λ into $\operatorname{Iso} B$, mapping Γ onto $\operatorname{Aut} B$.

By (6.5), if $\widetilde{A}(0) = 0$, $A_{12} = 0$; (6.3) implies then that A_{22} is a linear isometry of $\mathcal{K}$, *i.e.*, since $\dim_{\mathbf{C}} \mathcal{K} < \infty$, is a unitary operator in $\mathcal{K}$. Thus (6.4) yields $A_{21} = 0$, and, by (6.2), A_{11} is a linear isometry in $\mathcal{H}$. By Theorem I, that proves that, if $F \in (\operatorname{Iso} B)_0$, then $F \in \widetilde{\Lambda}$, the image of Λ by the map $A \to \widetilde{A}$. Since $\widetilde{\Lambda}$ contains $\operatorname{Aut} B$ [1] which acts transitively on B, a standard argument shows that $\widetilde{\Lambda} = \operatorname{Iso} B$, proving thereby

THEOREM II. *If* $\dim_{\mathbf{C}} \mathcal{K} < \infty$, *the map* $A \to \widetilde{A}$ *is a surjective homomorphism of* Λ *onto* $\operatorname{Iso} B$.

As a consequence, the results established in [8] for $\widetilde{\Lambda}$ hold for the entire semigroup $\operatorname{Iso} B$. For example, by Propositions 3.7 and 3.8 of [8], every $F \in \operatorname{Iso} B$ is the restriction to B of a weakly continuous map $\hat{F}: \overline{B} \to \overline{B}$. The Schauder-Tychonoff theorem implies then that $\hat{F}$ has some fixed point in $\overline{B}$.

Furthermore the strongly continuous linear semigroups in Λ constructed in [8] yield all the one-parameter semigroups in $\operatorname{Iso} B$ which are continuous for the strong topology in $\mathcal{L}(\mathcal{K}, \mathcal{H})$.

Proposition 4.2 of [8] yields

PROPOSITION 6. *Let* D *be a domain in* $\mathbf{C}$. *If* $\dim_{\mathbf{C}} \mathcal{K} < \infty$, *every holomorphic map* $f: D \times B \to B$ *for which* $g(z, \cdot) \in \operatorname{Iso} B$ *for all* $z \in D$, *is independent of* z.

Questa *Nota* è dedicata alla memoria di Edoardo Amaldi.

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