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Mathematical study of an evolution problem describing the thermomechanical process in shape memory alloys

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Equazioni a derivate parziali. — *Mathematical study of an evolution problem describing the thermo-mechanical process in shape memory alloys.* Nota (*) di PIERLUIGI COLLI, presentata dal Socio E. MAGENES.

ABSTRACT. — In this paper we prove existence, uniqueness, and continuous dependence for a one-dimensional time-dependent problem related to a thermo-mechanical model of structural phase transitions in solids. This model assumes the free energy depending on temperature, macroscopic deformation and also on the proportions of the phases. Here we neglect regularizing terms in the momentum balance equation and in the constitutive laws for the phase proportions.

KEY WORDS: Shape memory alloys; Thermo-mechanical model; Initial-boundary value problem; Existence and uniqueness.

RIASSUNTO. — *Studio matematico di un problema d'evoluzione che descrive il processo termomeccanico nelle leghe a memoria di forma.* In questa *Nota* si provano esistenza, unicità e dipendenza continua per un problema unidimensionale e dipendente dal tempo, relativo a un modello termomeccanico di transizioni di fase strutturali nei solidi. In questo modello si suppone che l'energia libera dipenda da temperatura, deformazioni macroscopiche e anche dalle proporzioni di fase. Si trascurano termini regolarizzanti nella equazione di bilancio del momento e nella legge costitutiva delle proporzioni di fase.

1. INTRODUCTION

There are various metallic alloys and other materials like polymers which present a peculiar and surprising physical property: after a permanent mechanical (*plastic*) deformation (for instance by traction), it is possible to recover their original shape with a suitable thermal treatment, that is, by heating or cooling. This phenomenon is known as *shape memory* and can be considered as the effect of a structural *austenite-martensite phase transition* [1, 3, 8, 11, 12].

Dealing with a *microscopic scale*, these alloys are composed by a mixture of crystals in austenitic or martensitic variants. The austenite phase is homogeneous and exhibits higher symmetry, while the martensite phase presents less symmetry and is internally twinned, that is, is organized into several *variants* related by symmetry. As an example of austenite-martensite phase transition, one could take the cubic-tetragonal transition in Indium-Thallium alloys (see, *e.g.*, [3]).

At a *macroscopic scale*, one can suppose that the phases coexist at each point with appropriate proportions. Also, even if several martensitic variants may appear in these materials [1, 3], the assumption of just *two low-symmetric phases* and a *single high-symmetric phase* is sufficient to give an exhaustive description of the shape memory behaviour. Taking into account these facts, Frémond [9] proposed a mathematical model to study the thermo-mechanical evolution of a three-dimensional *shape memory body*. As other macroscopic models dealing with phase transitions in these alloys (see, *e.g.*, [2, 3, 12, 14, 15] and references therein), also Frémond's one [7, 9] assumes the temperature and the macroscopic deformation as state variables. But, in addition, this last model is based on simple and

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classical expressions for the free energies of the phases and takes the volumetric proportions of austenite and martensite as thermodynamic quantities. Then the total free energy is obtained summing up the weighted free energies of the different phases and a *mixture* free energy (analogous to those of, *e.g.*, [4, 10]), which has the form of an indicator function and expresses compatibility conditions for the proportions. From constitutive laws and equilibrium equations one obtains *balance equations* for *energy* and *momentum* coupled with an *evolution variational inequality* for the *phase proportions*. Besides the quasi-static situation is considered (*i.e.* the momentum equation is in stationary form), and deformations are assumed to be small. For the detailed presentation of the model we refer to [7], where the related mathematical problem has been shown to be well-posed. A numerical approach to this problem is given in [16].

An important aspect of Frémond's model is that it takes into account the mechanical actions exerted on surfaces following the *second gradient theory*, restricting however to diagonal components depending on the trace of the *strain tensor* (see [7]). This provides a *fourth-order term* in the momentum balance equation which considerably helps in proving existence and uniqueness of the solution in the multi-dimensional problem [6, 7]. Another possibility of regularization is the introduction of a small *diffusion term* in the variational inequality regarding the phase proportions (see the final remark of [7]): that can be justified from the physical viewpoint by assuming some diffusive effects due to composition changes in the alloy. Concerning the regularizations just mentioned, we quote [13], where both ones are retained in order to treat two additional highly nonlinear coupling terms in the energy balance equations.

In the present paper we study the problem obtained by Frémond's model *without any fourth-order term* in the momentum balance equation. Moreover the variational inequality for phase proportions *does not include diffusion terms*. We consider the one-dimensional case and show that the reduced problem is still well-posed: unfortunately we are not able to reproduce the same result in two dimensions of space.

Let us briefly recall the new problem (always referring to [7] for the details). Consider a *shape memory wire* occupying a space interval $[0, L]$ at each point $x \in [0, L]$ and at each time $t \in [0, T]$ ($L, T > 0$ given). The physical unknowns are the *absolute temperature* ϑ, the variables χ_1 and χ_2 (obtained from the *local proportions* of the two martensitic variants by means of a rotation), and the *longitudinal displacement* u. We denote by λ, k, k_1, k_2, ϑ^* some positive constants (cf. [7, 9] for the physical meanings) and let $F, G:]0, L[\to \mathbf{R}$ represent functions proportional to the body forces and to the distributed heat sources, respectively. Then the energy and momentum balance equations and the constitutive law for the phase proportions are respectively:

$$(1.1) \qquad \frac{\partial}{\partial t}(\vartheta - \lambda\chi_1) - \frac{\partial^2 \vartheta}{\partial x^2} = F \qquad \text{in } Q :=]0, L[\times]0, T[,$$

$$(1.2) \qquad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \alpha(\vartheta)\chi_2\right) + G = 0 \qquad \text{in } Q,$$

$$(1.3) \qquad k\frac{\partial}{\partial t}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \begin{pmatrix}k_1(\vartheta - \vartheta^*)\\ k_2\,\alpha(\vartheta)u_x\end{pmatrix} + \partial I_{\tilde{K}}(\chi_1, \chi_2) \ni \begin{pmatrix}0\\ 0\end{pmatrix} \qquad \text{in } Q.$$

Here $\alpha(\vartheta)$ is a non negative and non increasing function which is proportional to the thermal expansion coefficient and vanishes for any temperature larger than a critical temperature ϑ_c, with $\vartheta_c > \vartheta^*$ (see [9]). Besides $\partial I_{\tilde{K}}$ denotes the subdifferential of the indicator function $I_{\tilde{K}}$ of a bounded convex set $\tilde{K} \subset \mathbf{R}^2$ containing the *admissible* (χ_1, χ_2), namely

$$I_{\tilde{K}}(\chi_1, \chi_2) = \begin{cases} 0 & \text{if } (\chi_1, \chi_2) \in \tilde{K}, \\ +\infty & \text{if } (\chi_1, \chi_2) \notin \tilde{K}. \end{cases}$$

The formulation of the problem has to be completed with suitable boundary and initial conditions. According to the positions of [6, 7], we prescribe:

$$-\vartheta_x(0,t) + b_0 \vartheta(0,t) = f_0(t) \qquad \text{for } t \in]0,T[, \tag{1.4}$$

$$\vartheta_x(L,t) + b_L \vartheta(L,t) = f_L(t) \qquad \text{for } t \in]0,T[, \tag{1.5}$$

$$u(0,t) = 0 \qquad \text{for } t \in]0,T[, \tag{1.6}$$

$$u_x(L,t) + \alpha(\vartheta(L,t))\, \chi_2(L,t) = g(t) \qquad \text{for } t \in]0,T[, \tag{1.7}$$

$$\vartheta(x,0) = \vartheta^0(x), \quad \chi_1(x,0) = \chi_1^0(x), \quad \chi_2(x,0) = \chi_2^0(x) \qquad \text{for } x \in]0,L[, \tag{1.8}$$

where b_0, b_L denote positive constants and f_0, f_L, g, ϑ^0, χ_1^0, χ_2^0 are given functions whose regularity will be specified later.

In the next section we give a variational formulation of (1.1-8) and we show the existence and uniqueness of the solution of this problem by using the Contraction Mapping Principle. Section 3 is devoted to prove continuous dependence on the data, stability, and regularity results.

2. Existence and uniqueness of the solution

First we want to give a precise formulation of the problem (1.1-8). Then we introduce the following Hilbert spaces: $H := L^2(0,L)$, $W := H^1(0,L)$, $V := \{v \in W : v(0) = 0\}$. As usual, we identify H with its dual space H'. We recall that $V \subset W \subset C^0([0,L])$: since the functions of V vanish in 0, we can take the norm

$$\|v\|_V := \left\{ \int_0^L |v_x(x)|^2\, dx \right\}^{1/2},$$

for $v \in V$. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing either between W' and W, or between V' and V, and by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively the scalar product and the norm in H. Let $\tilde{K}$ be a bounded closed convex subset of $\mathbf{R}^2$ and let $K := \{(\gamma_1, \gamma_2) \in H^2 : (\gamma_1, \gamma_2) \in \tilde{K}$ *a.e.* in $]0,L[\}$.

Since $\tilde{K}$ is bounded in $\mathbf{R}^2$, it follows that $K \subset (L^\infty(0,L))^2$ and there is a constant $C_K > 0$ such that for any $(\gamma_1, \gamma_2) \in K$

$$\|\gamma_2\|_{L^\infty(0,L)} \leq C_K. \tag{2.1}$$

Next, we list the assumptions on the data.

$$\alpha \in W^{1,\infty}(\mathbf{R}), \tag{2.2}$$

$$F \in L^2(0,T;W'), \qquad f_0, f_L \in L^2(0,T), \tag{2.3}$$

$$G \in L^\infty(0,T;H), \qquad g \in L^\infty(0,T), \tag{2.4}$$

$$\vartheta^0 \in H, \qquad (\chi_1^0, \chi_2^0) \in K. \tag{2.5}$$

Here is the variational formulation of problem (1.1-8).

Problem (P). *Find* $\vartheta \in L^2(0,T;W) \cap H^1(0,T;W')$, $u \in L^\infty(0,T;V)$, $\chi_1, \chi_2 \in H^1(0,T;H)$ *satisfying* $(\chi_1, \chi_2) \in K$,

$$\left\langle \frac{d}{dt}(\vartheta - \lambda\chi_1), \varphi \right\rangle + (\vartheta_x, \varphi_x) + \{b_L \vartheta(L,\cdot) - f_L\}\varphi(L) + \\ + \{b_0 \vartheta(0,\cdot) - f_0\}\varphi(0) = \langle F, \varphi \rangle \qquad \forall \varphi \in W, \tag{2.6}$$

$$(u_x, v_x) + (\alpha(\vartheta)\chi_2, v_x) = (G, v) + gv(L) \qquad \forall v \in V, \tag{2.7}$$

$$\sum_{i=1}^{2} k\left(\frac{d\chi_i}{dt}, \chi_i - \gamma_i\right) + k_1(\vartheta - \vartheta^*, \chi_1 - \gamma_1) + \\ + k_2(\alpha(\vartheta)u_x, \chi_2 - \gamma_2) \leqslant 0 \qquad \forall (\gamma_1, \gamma_2) \in K, \tag{2.8}$$

a.e. *in* $]0,T[$, *and such that*

$$\vartheta(\cdot,0) = \vartheta^0, \quad \chi_1(\cdot,0) = \chi_1^0, \quad \chi_2(\cdot,0) = \chi_2^0, \qquad \text{a.e. } in\ ]0,L[. \tag{2.9}$$

Remark 1. Note that all the equations of Problem (P) make sense. Indeed, observe for instance that in (2.7) and (2.8) the terms containing nonlinearities are meaningful thanks to (2.2) (see (2.1) too), that is $\alpha(\vartheta)$ (as well as χ_2) is essentially bounded in $Q =]0,L[\times]0,T[$. Besides one can easily see, also by interpolation, that ϑ, χ_1, $\chi_2 \in C^0([0,T];H)$: then from (2.5) it follows that (2.9) has a meaning.

Theorem 1. *Problem* (P) *has one and only one solution.*

The remaining part of this section is devoted to prove the theorem. To this aim, we introduce the space $Y := C^0([0,T];H^2)$ and the closed convex sets

$$Y_0 := \{(\gamma_1, \gamma_2) \in Y : \gamma_1(\cdot,0) = \chi_1^0,\ \gamma_2(\cdot,0) = \chi_2^0\}, \tag{2.10}$$

$$X := \{(\gamma_1, \gamma_2) \in Y_0 : (\gamma_1(\cdot,t), \gamma_2(\cdot,t)) \in K \ \ \forall t \in [0,T]\}, \tag{2.11}$$

Proof. We follow [6] in parts. Let us briefly sketch our procedure. Fix any element $(\chi_1, \chi_2) \in X$. First we find a weak solution $\vartheta = \Theta(\chi_1)$ of (2.6) satisfying the related initial condition present in (2.9). Next, we solve (2.7) (where $\vartheta = \Theta(\chi_1)$) getting a function $u = U(\vartheta, \chi_2)$. Finally, utilizing the already found ϑ and u, we look for a solution of (2.8) subject to the initial conditions in (2.9): this determines a *new* pair $(\chi_1, \chi_2) = (\mathcal{X}_1, \mathcal{X}_2)(\vartheta, u) \in X$. Thus we construct an operator $A : X \to X$. We want to show that, for an integer m sufficient large, A^m is a contraction mapping in X: then Problem (P) will have a unique global solution.

For the reader's convenience we split the procedure into steps.

STEP 1. For any $\chi_1 \in C^0([0,T];H)$ there is one and only one $\vartheta = \Theta(\chi_1)$ such that, setting $I_\vartheta(\cdot,t) = \int_0^t \vartheta(\cdot,s)ds$ for *a.e.* $t \in]0,T[$, ϑ, I_ϑ satisfy

$$\vartheta \in L^2(0,T;H), \quad I_\vartheta \in L^\infty(0,T;W), \tag{2.12}$$

$$\langle(\vartheta - \lambda\chi_1)(\cdot,t),\varphi\rangle + \left(\frac{\partial I_\vartheta}{\partial x}(\cdot,t),\varphi_x\right) + \left\{b_L I_\vartheta(L,\cdot) - \int_0^t f_L(s)ds\right\}\varphi(L) + \tag{2.13}$$
$$+\left\{b_0 I_\vartheta(0,\cdot) - \int_0^t f_0(s)ds\right\}\varphi(0) = \left\langle \vartheta^0 - \lambda\chi_1^0 + \int_0^t F(\cdot,s)ds, \varphi\right\rangle \qquad \forall\varphi \in W,$$

for *a.e.* $t \in]0,T[$. Moreover, if $\chi_1 \in H^1(0,T;H)$, then $\vartheta \in L^2(0,T;W) \cap H^1(0,T;W')$ and solves (2.6).

For the proof of the above statement one can see Lemma 1 of [6]. It should also be noted that if $(\chi_1,\chi_2) \in Y_0$, then (cf. (2.10) and (2.13)) $\vartheta(\cdot,0) = \vartheta^0$. Next, one can easily check that from (2.13) it follows that for any χ_1, $\tilde{\chi}_1 \in C^0([0,T];H)$ we have

$$\int_0^t \|(\Theta(\chi_1) - \Theta(\tilde{\chi}_1))(\cdot,s)\|^2 ds \leqslant \lambda^2 \int_0^t \|(\chi_1 - \tilde{\chi}_1)(\cdot,s)\|^2 ds \qquad \forall t \in [0,T]. \tag{2.14}$$

STEP 2. It is a standard matter to see (applying for instance the Lax-Milgram Lemma) that for any $\vartheta \in L^2(0,T;H)$, $\chi_2 \in L^\infty(0,T;H)$ there exists a unique $u = U(\vartheta,\chi_2) \in L^\infty(0,T;V)$ satisfying (2.7) *a.e.* in $]0,T[$. Indeed, it suffices to note that, by (2.2) and (2.4), $(G,v) + g\,v(L) - (\alpha(\vartheta)\chi_2, v_x)$ is a linear and continuous operator on V for *a.e.* $t \in]0,T[$. Moreover, taking first an arbitrary $v \in H_0^1(0,L)$ and integrating by parts in (2.7), and then, with a standard procedure, recovering the boundary condition (1.7) in the sense of traces, we obtain

$$u_x(x,t) = \int_x^L G(\xi,t)d\xi + g(t) - \alpha(\vartheta(x,t))\chi_2(x,t) \qquad \text{for } a.e.\ (x,t) \in Q. \tag{2.15}$$

Let now $(\chi_1,\chi_2) \in X$ so that χ_2 satisfy (2.1) *a.e.* in $]0,T[$. From (2.15), (2.2), and (2.4) it follows that there is a constant C_1, depending only on L, C_K, $\|G\|_{L^\infty(0,T;H)}$, $\|g\|_{L^\infty(0,T)}$, and $\|\alpha\|_{L^\infty(R)}$, such that

$$\|u_x\|_{L^\infty(Q)} \leqslant C_1. \tag{2.16}$$

Next, by using (2.7) it is straightforward to show that there is a constant C_2 such that for any ϑ, $\tilde{\vartheta} \in L^2(0,T;H)$, (χ_1,χ_2), $(\tilde{\chi}_1,\tilde{\chi}_2) \in X$ one has

$$(2.17) \qquad \int_0^t \|(U(\vartheta,\chi_2) - U(\tilde{\vartheta},\tilde{\chi}_2))(\cdot,s)\|_V^2\, ds \leqslant$$

$$\leqslant C_2 \int_0^t \{\|(\vartheta - \tilde{\vartheta})(\cdot,s)\|^2 + \|(\chi_2 - \tilde{\chi}_2)(\cdot,s)\|^2\} ds \qquad \forall t \in [0,T],$$

where, for instance, $C_2 = 2\|\alpha\|^2_{W^{1,\infty}(\mathbf{R})}\{1 + (C_K)^2\}$ (cf. (2.1) and (2.2)).

Step 3. For any $\vartheta \in L^2(0,T;H)$, $u \in L^2(0,T;V)$ there exists one and only one pair $\chi_1 = \mathcal{X}_1(\vartheta,u)$, $\chi_2 = \mathcal{X}_2(\vartheta,u)$ such that $(\chi_1,\chi_2) \in X \cap H^1(0,T;H^2)$ and (2.8) holds *a.e.* in $]0,T[$. Moreover there exists a constant C_3, depending only on k, k_1, k_2, C_1, and $\|\alpha\|_{W^{1,\infty}(\mathbf{R})}$, such that for any ϑ, $\tilde{\vartheta} \in L^2(0,T;H)$ and any u, $\tilde{u} \in L^2(0,T;V)$ satisfying (2.16) we have

$$(2.18) \qquad \|((\mathcal{X}_1,\mathcal{X}_2)(\vartheta,u) - (\mathcal{X}_1,\mathcal{X}_2)(\tilde{\vartheta},\tilde{u}))(\cdot,t)\|^2_{H^2} = \sum_{i=1}^{2} \|(\mathcal{X}_i(\vartheta,u) - \mathcal{X}_i(\tilde{\vartheta},\tilde{u}))(\cdot,t)\|^2 \leqslant$$

$$\leqslant C_3\, t \int_0^t \{\|(\vartheta - \tilde{\vartheta})(\cdot,s)\|^2 + \|(u - \tilde{u})(\cdot,s)\|_V^2\} ds \qquad \forall t \in [0,T].$$

This statement is proved in Lemma 3 of [6].

Step 4 (end of the proof). We define the operator $A: X \to X$ in the following way: $A(\chi_1,\chi_2) = (\mathcal{X}_1,\mathcal{X}_2)(\Theta(\chi_1), U(\Theta(\chi_1),\chi_2))$, where the operators Θ, U, $(\mathcal{X}_1,\mathcal{X}_2)$ have been defined in the previous steps. Remark that for $(\chi_1,\chi_2) \in X$ in addition we have $A(\chi_1,\chi_2) \in H^1(0,T;H^2)$ (see Step 3).
Making use of (2.18), (2.17) and (2.14), it is easy to see that there is a constant C such that for any (χ_1,χ_2), $(\tilde{\chi}_1,\tilde{\chi}_2) \in X$ one has

$$(2.19) \qquad \|(A(\chi_1,\chi_2) - A(\tilde{\chi}_1,\tilde{\chi}_2))(\cdot,t)\|^2 \leqslant C t^2 \|(\chi_1,\chi_2) - (\tilde{\chi}_1,\tilde{\chi}_2)\|_Y^2 \qquad \forall t \in [0,T],$$

where, for instance, $C = C_3\{C_2 + (1 + C_2)\lambda^2\}$. Next, applying the same argument to $A(\chi_1,\chi_2)$, $A(\tilde{\chi}_1,\tilde{\chi}_2)$ and accounting for (2.19), we find that

$$\|(A^2(\chi_1,\chi_2) - A^2(\tilde{\chi}_1,\tilde{\chi}_2))(\cdot,t)\|^2_{H^2} \leqslant \{C^2 t^4/3\}\|(\chi_1,\chi_2) - (\tilde{\chi}_1,\tilde{\chi}_2)\|_Y^2 \qquad \forall t \in [0,T].$$

Hence, by induction,

$$\|A^m(\chi_1,\chi_2) - A^m(\tilde{\chi}_1,\tilde{\chi}_2)\|_Y^2 \leqslant C^m T^{2m}(1\cdot 3 \cdots (2m-1))^{-1}\|(\chi_1,\chi_2) - (\tilde{\chi}_1,\tilde{\chi}_2)\|_Y^2$$

$$\forall m \in \mathbb{N}, \qquad \forall (\chi_1,\chi_2), (\tilde{\chi}_1,\tilde{\chi}_2) \in X.$$

Thus, provided that m is large enough, A^m is a contraction mapping in X and the theorem is completely proved.

Remark 2. The crucial part of our proof is Step 2: otherwise the procedure follows directly from those of [6, 7]. Thanks to the fact that the problem is one-dimensional, we are able to state (2.16) (consequence of (2.15)) even if (1.2) is just a second-order equation in space. As one can easily see (cf. also [7]), the same argument seems not applicable in two dimensions of space (where (1.2) will become a system) in order to recover L^∞ regularity for $\operatorname{div} \boldsymbol{u}$.

3. Other results

The first part of this section is devoted to the proof of the continuous dependence of the solutions of Problem (P) with respect to the data. To this end consider another set of given functions $\{\tilde{\alpha}, \tilde{F}, \tilde{f}_0, \tilde{f}_L, \tilde{G}, \tilde{g}, \tilde{\vartheta}^0, (\tilde{\chi}_1^0, \tilde{\chi}_2^0)\}$ satisfing (2.2-5), as well as $\{\alpha, F, f_0, f_L, G, g, \vartheta^0, (\chi_1^0, \chi_2^0)\}$. Denote by $\tilde{\vartheta}, \tilde{u}, \tilde{\chi}_1, \tilde{\chi}_2$ and $\vartheta, u, \chi_1, \chi_2$ the solutions of Problem (P) corresponding to the two sets of data. We introduce a constant C_4 which will be useful in the statement of the result:

$$(3.1) \qquad C_4 := \max\{\|\alpha\|_{W^{1,\infty}(\mathbf{R})},\ \|\tilde{\alpha}\|_{W^{1,\infty}(\mathbf{R})},\ \| |G| + |\tilde{G}| \|_{L^\infty(0,T;H)},\ \| |g| + |\tilde{g}| \|_{L^\infty(0,T)} \}.$$

Theorem 2. *There exists a positive constant C_5, depending only on L, λ, b_0, b_L, C_K, k, k_1, k_2, C_4, such that*

$$(3.2) \qquad \|\vartheta - \tilde{\vartheta}\|_{L^2(0,T;H)} + \max_{0\le t\le T} \left\| \int_0^t (\vartheta - \tilde{\vartheta})(\cdot, s)\,ds \right\|_W + \|u - \tilde{u}\|_{L^2(0,T;V)} +$$

$$+\|(\chi_1, \chi_2) - (\tilde{\chi}_1, \tilde{\chi}_2)\|_{L^\infty(0,T;H^2)} \le C_5 \{\|\alpha - \tilde{\alpha}\|_{L^\infty(\mathbf{R})} + \|F - \tilde{F}\|_{L^2(0,T;W')} +$$

$$+\| f_0 - \tilde{f}_0 \|_{L^2(0,T)} + \| f_L - \tilde{f}_L \|_{L^2(0,T)} + \|G - \tilde{G}\|_{L^2(0,T;H)} +$$

$$+\|g - \tilde{g}\|_{L^2(0,T)} + \|\vartheta^0 - \tilde{\vartheta}^0\|_H + \|(\chi_1^0, \chi_2^0) - (\tilde{\chi}_1^0, \tilde{\chi}_2^0)\|_{H^2}\}.$$

Proof. We set $\hat{\vartheta} = \vartheta - \tilde{\vartheta}$, $\hat{u} = u - \tilde{u}$, $\hat{\chi}_i = \chi_i - \tilde{\chi}_i$, $i = 1, 2$. Still for the sake of brevity we denote by S the quantity multiplying C_5 in the right hand side of (3.2). Besides, let $b > 0$ be a constant satisfying

$$(3.3) \qquad b\|\varphi\|_W^2 \le \|\varphi_x\|^2 + b_0\varphi^2(0) + b_L\varphi^2(L) \qquad \forall \varphi \in W.$$

Obviously b depends also on L. Consider now (2.13) for ϑ and $\tilde{\vartheta}$, take the difference and choose $\varphi = \hat{\vartheta}$. Then we integrate in time from 0 to $t \in [0, T]$ and estimate each term of the right hand side. Taking into account (2.3), (2.5), and (3.3), integrating by parts in time the terms containing $F - \tilde{F}$, $f_0 - \tilde{f}_0$, $f_L - \tilde{f}_L$, and moving on the left hand side each quantity proportional (by a small positive number) to $\int_0^t \|\hat{\vartheta}(\cdot, s)\|^2\,ds$ and to $\left\| \int_0^t \hat{\vartheta}(\cdot, s)\,ds \right\|_W^2$, we obtain

$$(3.4) \qquad \frac{1}{2}\int_0^t \|\hat{\vartheta}(\cdot, s)\|^2\,ds + \frac{b}{4}\left\| \int_0^t \hat{\vartheta}(\cdot, s)\,ds \right\|_W^2 \le \{2T(1+\lambda^2) + 3(T+1)\,C_6\}\,S^2 +$$

$$+\frac{3}{2}\lambda^2 \int_0^t \|\hat{\chi}_1(\cdot, s)\|^2\,ds + \int_0^t \left\| \int_0^s \hat{\vartheta}(\cdot, \tau)\,d\tau \right\|_W^2 ds \qquad \forall t \in [0, T],$$

where the constant C_6 depends only on L, b. Next, we apply the same procedure to (2.7) by choosing $v = \hat{u}$. Due to the nonlinearity, here we have to sum and subtract some auxiliary terms. Using (2.1), (2.4), (3.1), and taking into account that

$$\int_0^T \|(\alpha(\vartheta)\chi_2 - \tilde{\alpha}(\vartheta)\chi_2)(\cdot, t)\|^2 dt \leqslant TL(C_K)^2 \|\alpha - \tilde{\alpha}\|^2_{L^\infty(\mathbf{R})}, \tag{3.5}$$

we get

$$\frac{1}{2}\int_0^t \|\hat{u}(\cdot, s)\|_V^2 ds \leqslant \{2(1 + L + TL(C_K)^2)\} S^2 + C_7 \int_0^t \|\hat{\vartheta}(\cdot, s)\|^2 ds + \tag{3.6}$$
$$+ \frac{3}{2}(C_4)^2 \int_0^t \|\hat{\chi}_2(\cdot, s)\|^2 ds \qquad \forall t \in [0, T],$$

where, for instance, $C_7 = 3(C_K C_4)^2/2$. In (2.8) we take $\gamma_i = \tilde{\chi}_i$ for χ_i, and $\gamma_i = \chi_i$ for $\tilde{\chi}_i$, $i = 1, 2$. Then we add the inequalities and integrate in time using (2.9). Also here we have to sum and subtract auxiliary quantities in order to estimate the right hand side. Recalling (3.1) and (2.15-16), note that there is a constant C_8, depending only on L, C_K, C_4, such that $\max\{\|u_x\|_{L^\infty(Q)}, \|\tilde{u}_x\|_{L^\infty(Q)}\} \leqslant C_8$. From this inequality, (2.5), and (3.5) (with χ_2, C_K replaced by u_x, C_8 respectively), it follows that

$$\frac{k}{2}\|(\hat{\chi}_1, \hat{\chi}_2)(\cdot, t)\|^2_{H^2} \leqslant \left\{ \frac{k}{2} + LT \left| \frac{C_8}{2C_4} \right|^2 \right\} S^2 + \tag{3.7}$$
$$+ (k_1 + k_2 C_4)^2 \int_0^t \|(\hat{\chi}_1, \hat{\chi}_2)(\cdot, s)\|^2_{H^2} ds + C_9 \int_0^t \|\hat{\vartheta}(\cdot, s)\|^2 ds + \frac{1}{4}\int_0^t \|\hat{u}(\cdot, s)\|^2_V ds$$

for any $t \in [0, T]$, where, for instance, $C_9 = \{1 + (C_8)^2\}/4$. Then we multiply (3.4) by $4(C_7 + C_9)$ and add (3.6) and (3.7), obtaining

$$(C_7 + C_9)\left\{ \int_0^t \|\hat{\vartheta}(\cdot, s)\|^2 ds + b \left\| \int_0^t \hat{\vartheta}(\cdot, s) ds \right\|^2_W \right\} + \frac{1}{4}\int_0^t \|\hat{u}(\cdot, s)\|^2_V ds + \tag{3.8}$$
$$+ \frac{k}{2}\|(\hat{\chi}_1, \hat{\chi}_2)(\cdot, t)\|^2_{H^2} \leqslant C_{10}\left\{ S^2 + \int_0^t \left\| \int_0^s \hat{\vartheta}(\cdot, \tau) d\tau \right\|^2_W ds + \int_0^t \|(\hat{\chi}_1, \hat{\chi}_2)(\cdot, s)\|^2_{H^2} ds \right\} \quad \forall t \in [0, T],$$

where the constant C_{10} has the same dependences as C_5. Finally, by applying to (3.8) the Gronwall Lemma (see, *e.g.*, [5, p. 156]), the estimate (3.2) easily follows.

The next result provides a stability estimate for the solution of Problem (P).

THEOREM 3. *Let ϑ, u, χ_1, χ_2 be the solution of Problem (P) corresponding to the data in (2.2-5). Then there exists a constant C_{11}, depending on T and the data, such that*

$$\|\vartheta\|_{C^0([0,T];H) \cap L^2(0,T;W)} + \|u\|_{L^\infty(0,T;W^{1,\infty}(0,L))} + \|(\chi_1, \chi_2)\|_{H^1(0,T;H^2)} \leqslant C_{11}. \tag{3.9}$$

PROOF. We start by dealing with u. From (2.15-16) and (1.6) it follows that

$$\|u\|_{L^\infty(0,T;W^{1,\infty}(0,L))} \leq C_1(1+L), \tag{3.10}$$

independently of ϑ and χ_2. Then, (2.8-9) and (2.5) (cf. also (1.3)) yield the following estimate (see, *e.g.*, [5, Thm. 3.6, p. 73])

$$k^2 \sum_{i=1}^{2} \int_0^t \left\| \frac{\partial \chi_i}{\partial t}(\cdot, s) \right\|^2 ds \leq (k_1)^2 \int_0^t \|(\vartheta - \vartheta^*)(\cdot, s)\|^2 ds + (k_2 C_1 \|\alpha\|_{L^\infty(\mathbf{R})})^2 Lt \tag{3.11}$$

for any $t \in [0, T]$. Now we take $\varphi = \vartheta$ in (2.6), integrate it in time from 0 to $t \in [0, T]$, and estimate each term of the right hand side. Using also (3.3), we obtain

$$\begin{aligned} \frac{1}{2}\|\vartheta(\cdot, t)\|^2 + \frac{b}{2}\int_0^t \|\vartheta(\cdot, s)\|_W^2 ds \leq \frac{1}{2}\|\vartheta^0\|^2 + \frac{k^2}{2}\int_0^t \left\| \frac{\partial \chi_1}{\partial t}(\cdot, s) \right\|^2 ds + \\ + \frac{\lambda^2}{2k^2}\int_0^t \|\vartheta(\cdot, s)\|^2 ds + C_{12} \qquad \forall t \in [0, T], \end{aligned} \tag{3.12}$$

where C_{12} depends on L, b, $\|F\|_{L^2(0,T;W')}$, $\|f_0\|_{L^2(0,T)}$, $\|f_L\|_{L^2(0,T)}$. We sum up (3.11) and (3.12), then apply the Gronwall Lemma: taking into account also (3.10), the theorem is proved.

REMARK 3. As we could expect (in fact (3.9) is a stability estimate), the constant C_{11} of (3.9) does not depend on $\|\alpha'\|_{L^\infty(\mathbf{R})}$: just $\|\alpha\|_{L^\infty(\mathbf{R})}$ is concerned (cf. (2.16), (3.10), (3.11)).

We conclude this section with a regularity result for the solution ϑ, u, χ_1, χ_2 of (P).

THEOREM 4. *Assume (2.2-5) hold and moreover let* $\vartheta^0 \in W$, $f_0, f_L \in W^{1,1}(0,T)$, $F = F_1 + F_2$, *where* $F_1 \in W^{1,1}(0,T;W')$ *and* $F_2 \in L^2(0,T;H)$. *Then* $\vartheta \in H^1(0,T;H) \cap L^\infty(0,T;W)$. *If, in addition,* $G \in H^1(0,T;L^1(0,L))$ *and* $g \in H^1(0,T)$, *then* $u \in H^1(0,T;V)$.

PROOF. The first part of the statement can be shown by standard arguments: provided a suitable regularization of (2.6), we are allowed to take $\varphi = \vartheta_t$ in (2.6), then we integrate in time, estimate the right hand side by integrating by parts some terms, and finally utilize the Gronwall Lemma. The additional regularity of u follows easily, for instance, from (2.15), differentiating it with respect to time, and from (1.6).

REMARK 4. Concerning the physical point of view, the coefficient k appearing in (1.3) and (2.8) represents a viscosity rate. Observe that the constants C_5 and C_{11} of the continuous dependence and stability estimates blow up as k goes to zero. Nevertheless, it would be very interesting for the applications (cf., *e.g.*, [7, 9, 10]) to analyse the asymptotic behaviour of Problem (P) when $k \searrow 0$ and study the limit problem: for this one, the existence of a solution seems to be already an intriguing open question.

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References

[1] M. Achenbach - I. Müller, *Model for shape memory.* J. Physique, C4 Suppl. 12, 43, 1982, 163-167.

[2] H. W. Alt - K. H. Hoffmann - M. Niezgódka - J. Sprekels, *A numerical study of structural phase transitions in shape memory alloys.* Preprint n. 90, Institut für Mathematik, Universität Augsburg, 1985.

[3] S. Antman - J. L. Ericksen - D. Kinderlehrer - I. Müller (eds.), *Metastability and incompletely posed problems.* Springer-Verlag, Berlin 1987.

[4] D. Blanchard - M. Frémond - A. Visintin, *Phase change with dissipation.* In: H. D. Buy - Q. S. Nguyen (eds.), *Thermomechanical coupling in solids.* North-Holland, Amsterdam 1987, 411-418.

[5] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland, Amsterdam 1973.

[6] P. Colli, *An evolution problem related to shape memory alloys.* In: J. F. Rodrigues (ed.), *Mathematical models for phase change problems.* International Series of Numerical Mathematics, 88, Birkhäuser Verlag, Basel 1989, 75-88.

[7] P. Colli - M. Frémond - A. Visintin, *Thermo-mechanical evolution of shape memory alloys.* Quart. Appl. Math., 48, 1990, 31-47.

[8] F. Falk, *Martensitic domain boundaries in shape-memory alloys as solitary waves.* J. Physique, C4 Suppl. 12, 43, 1982, 3-15.

[9] M. Frémond, *Matériaux a mémoire de forme.* C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre, 304, 1987, 239-244.

[10] M. Frémond - A. Visintin, *Dissipation dans le changement de phase. Surfusion. Changement de phase irréversible.* C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre, 301, 1985, 1265-1268.

[11] G. Herget - M. Müllner - G. Eckold - H. Tietze - W. Assmus, *Strukturelle phasenübergänge der formgedächtnislegierung NiTi.* Jahresbericht 1987 des Inst. für Kernphysik, Johann Wolfgang Göthe Universität, Frankfurt/Main 1987, 77.

[12] K. H. Hoffmann - M. Niezgódka, *Mathematical models of dynamical martensitic transformations in shape memory alloys.* Preprint n. 201, Institut für Mathematik, Universität Augsburg, 1990.

[13] K. H. Hoffmann - M. Niezgódka - S. Zheng, *Existence and uniqueness to an extended model of the dynamical developments in shape memory alloys.* Nonlinear Analysis, to appear.

[14] M. Niezgódka - J. Sprekels, *Existence of solutions for a mathematical model of structural phase transitions in shape memory alloys.* Math. Methods Appl. Sci., 10, 1988, 197-223.

[15] J. Sprekels - S. Zheng, *Global solutions of a Ginzburg-Landau theory for structural phase transitions in shape memory alloys.* Physica D 39, 1989, 59-76.

[16] T. Tiihonen, *A numerical approach to a shape memory model.* Preprint n. 98, Institut für Mathematik, Universität Augsburg, 1988.

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