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A Wiener type criterion for weighted quasiminima

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Calcolo delle variazioni. — *A Wiener type criterion for weighted quasiminima.* Nota (*) di SILVANA MARCHI, presentata dal Socio L. AMERIO.

ABSTRACT. — We prove a sufficient condition of continuity at the boundary for quasiminima of degenerate type. W. P. Ziemer stated a Wiener-type criterion for the quasiminima defined by Giaquinta and Giusti. In this paper we extend the result of Ziemer to the case of weighted quasiminima, the weight being in the A_2 class of Muckenhoupt.

KEY WORDS: Wiener criterion; Degenerate quasiminima; Weights in the A_2 class of Muckenhoupt.

RIASSUNTO. — *Un criterio di tipo Wiener per quasiminimi con peso.* Si prova una condizione sufficiente di continuità in punti di frontiera per quasiminimi di tipo degenere. W. P. Ziemer provò un criterio di tipo Wiener per i quasiminimi definiti da Giaquinta e Giusti. In questo articolo si estende il risultato di Ziemer al caso di quasiminimi con peso, il peso essendo nella classe A_2 di Muckenhoupt.

1. INTRODUCTION

In this paper we consider functionals of the form

$$J(u,\Omega) = \int_\Omega f(x,u,\nabla u)\,dx. \tag{1.1}$$

Here Ω is an open bounded connected set of $\mathbf{R}^N$, $N \geq 2$, $f: \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ is a Caratheodory function, namely measurable in x for every (z,p) and continuous in (z,p) for almost every $x \in \Omega$, $u: \Omega \to \mathbf{R}$ is a scalar function belonging to the weighted Sobolev space $H^1_{\text{loc}}(\Omega, w)$ (see [12] for the definition of weighted Sobolev spaces), the weight $w(x)$ is in the A_2 class of Muckenhoupt [8] (see §2 of this paper).

DEFINITION 1. (see [6]) We call a function $u \in H^1_{\text{loc}}(\Omega, w)$ a *sub Q-minimum* (*super Q-minimum*) for J if, for some constant $Q \geq 1$

$$J(u,K) \leq QJ(u+\varphi, K) \tag{1.2}$$

for every $\varphi \in H^1(\Omega, w)$, $\varphi \leq 0$ ($\varphi \geq 0$) with $\operatorname{supp}\varphi = K \subset \Omega$. A *Q-minimum* for J is both a sub and a super Q-minimum.

In order to prove a sufficient condition of continuity at the boundary for a Q-minimum we will suppose the following additional assumptions on f

$$|p|^2 - b|z|^2 - g(x) \leq f(x,z,p)/w(x) \leq \mu|p|^2 + b|z|^2 + g(x) \tag{1.3}$$

where $\mu \geq 1$, $b \in L^\infty_{\text{loc}}(\mathbf{R}^N)$, $g \in L^q(\Omega, w)$, $q > N$.

The weighted Choquet capacity used in this paper is defined on an arbitrary compact set Q by

$$C(Q) = \inf\left\{\int |\nabla v|^2 w \,|\, v \in C_0^\infty(\mathbf{R}^N), v \geq 1 \text{ on } E\right\}$$

(*) Pervenuta all'Accademia il 31 luglio 1990.

For G open, $C(G) = \sup_{Q \subset G} C(Q)$, and, for an arbitrary set E,

$$C(E) = \inf_{E \subset G} C(G) \tag{1.4}$$

Because of definition (1.4), C is an outer capacity.

If $x_0 \in \partial\Omega$ and $B_r(x_0)$ denote the ball of $\mathbf{R}^N$ centered at x_0 of radius r, we define

$$\delta(r) = C(B_r(x_0) - \Omega)/C(B_r(x_0)). \tag{1.5}$$

In this paper we prove the following result:

THEOREM. Let $u \in H^1_{loc}(\Omega, w)$ be a Q-minimum such that $u - \beta \in H^1_0(\Omega, w)$, where $\beta \in H^1(\mathbf{R}^N, w)$ is continuous on $\mathbf{R}^N - \Omega$. Let $x_0 \in \partial\Omega$.

Then there exists a positive constant c, depending only on the structure (2) and $\|u\|_{L_2(\Omega, w)}$, so that, if

$$\int_0^1 \exp(-c\delta(r)^{-2}) \frac{dr}{r} = +\infty \tag{1.6}$$

then

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = \beta(x_0) \tag{1.7}$$

holds.

In particular, if

(1.8) $\delta(r) \geq a > 0$ for all small r and some constant a then (1.7) holds.

The outline of the proof is that of Theorem 3.4 of [12] where the same result is stated in the case $w(x) = 1$. The main tool in the proof is the Harnack inequality for functions in weighted De Giorgi classes [9].

2. Further definitions and assumptions

We will suppose $w \in A_2$, that is w, $1/w \in L^1_{loc}(\mathbf{R}^N)$ and

$$\sup_B \left(\frac{1}{|B|}\int_B w\right)\left(\frac{1}{|B|}\int_B \frac{1}{w}\right) \leq c, \tag{2.1}$$

where B denote an arbitrary ball of $\mathbf{R}^N$. If $x_0 \in \partial\Omega$ is the point fixed in Theorem 1, and if $B_r = B_r(x_0)$, $r > 0$, then we will suppose, in addition

$$w(B_r)\, r^{-2} \leq c \qquad \text{for all small } r. \tag{2.2}$$

We will write $A(\ldots) \cong B(\ldots)$ if there exist two positive constants c' and c'' so that $c' B(\ldots) \leq A(\ldots) \leq c'' B(\ldots)$.

When $w \in A_2$, and E is a Suslin set, in particular a Borel set, one can prove (see [2, 1, 11]) that,

$$C(E) \cong \sup \{\|K(v, y)\|^{-2}_{L^2(\mathbf{R}^N, w)}\}, \tag{2.3}$$

where v is a positive measure concentrated on E, such that $1 \leq \|v\|_1 < +\infty$, $\|v\|_1$ being the total variation of v.

Here $K(x,y) = |x-y|^{1-N} w(y)^{-1}$ and $K(v,y) = \int K(x,y)\, dv(x)$.

3. Harnack inequality

We set $A^+(k,r) = B_r(x_0) \cap [u > k]$ and $A^-(k,r) = B_r(x_0) \cap [u < k]$.

The Q-minimum u is extended to all of $\mathbf{R}^N$ by $\beta(x)$ for $x \in \mathbf{R}^N - \Omega$.

Choose $k > \beta(3r) = \sup\{\beta(x) : x \in B_{3r}(x_0) - \Omega\}$ and define $u_k = (u-k)^+$, $\bar{u}_k(r) = \sup\{u_k(x) : x \in B_r(x_0)\}$, $\bar{u}(r) = \sup\{u(x) : x \in B_r(x_0)\}$.

The Harnack inequality at the boundary for sub Q-minima can now be stated.

Lemma. There exist some positive constants c and p_0 with $c = c(\|u\|_{L^2(\Omega, w)})$ such that

$$r^\alpha + \bar{u}_k(2r) - \bar{u}_k(r) \geq c \left(\int_{B(x_0, r)} [\bar{u}_k(2r) - \bar{u}_k(x)]^{p_0} w(x) \right)^{1/p_0}. \tag{3.1}$$

Proof. It is sufficient to show that $v(x) = \bar{u}_k(2r) - \bar{u}_k(x)$ belongs to the De Giorgi class $DG_2^-(\Omega_r, w)$, where $\Omega_r = \Omega \cup B_{3r}(x_0)$. Here we avoid the details of the demonstration that the interested reader can found in [13]. The thesis of Lemma then follows from [9], where we proved the Harnack inequality for functions in weighted De Giorgi classes.

4. Proof of the theorem

Let $\omega(r) = \overline{u_k}(2r) - \overline{u_k}(r) + r^\alpha$. Then

$$\int_0^1 \omega(r) \frac{dr}{r} < +\infty. \tag{4.1}$$

Suppose *ad absurdo* that there is a constant k such that

$$\limsup_{\substack{x \to x_0 \\ x \in \Omega}} u(x) > k > \beta(x_0). \tag{4.2}$$

If $k_j = \bar{u}(2r) - 2^{-j}\bar{u}_k(2r)$, $j = 1, 2, \ldots$, we obtain from Harnack inequality

$$\omega(r)^{p_0} \geq [2^{-j}\overline{u_k}(2r)]^{p_0} w(A^-(k_j, r))/w(B_r) \tag{4.3}$$

and then

$$\omega(r) \geq M \exp(-c\delta(r)^{-2}), \qquad M = M(\theta, p_0) > 0, \tag{4.4}$$

where θ is a positive constant such that $\overline{u_k}(2r) \geq \theta$ for all small r. This contradicts (4.1).

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