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Some results on the existence of geodesics in static Lorentz manifolds with singular boundary

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Analisi matematica. — *Some results on the existence of geodesics in static Lorentz manifolds with singular boundary.* Nota di Vieri Benci, Donato Fortunato e Fabio Giannoni, presentata (*) dal Corrisp. A. Ambrosetti.

Abstract. — In this *Note* we deal with the problem of the existence of geodesics joining two given points of certain non-complete Lorentz manifolds, of which the Schwarzschild spacetime is the simplest physical example.

Key words: Lorentz manifolds; Geodesics; Critical points.

Riassunto. — *Alcuni risultati sull'esistenza di geodetiche su varietà di Lorentz con bordo singolare.* In questa *Nota* trattiamo il problema dell'esistenza di geodetiche congiungenti due assegnati punti di certe varietà di Lorentz non complete, delle quali lo spazio-tempo di Schwarzschild è l'esempio fisico più semplice.

1. Introduction

Let $(\mathscr{L}, \langle\, , \rangle_L)$ be a Lorentz manifold, *i.e.* a smooth manifold equipped with a metric tensor $\langle\, , \rangle_L$ having index 1 (see *e.g.* [8]). (We recall that if the Lorentz manifold has dimension 4, it is called «spacetime»).

We consider Lorentz manifolds under assumptions that are satisfied for example by the Schwarzschild spacetime and the Reissner-Nordström spacetime (see *e.g.* [6]).

The Schwarzschild metric is the unique solution (up to isometric change of variables) of the Einstein equations in the empty space, when the curvature of the spacetime is produced by a single, static, spherically symmetric massive body.

Using polar coordinates this metric can be given in the form:

$$ds^2 = \beta(r)^{-1}\, dr^2 + r^2\, d\Omega^2 - c^2 \cdot \beta(r)\, dt^2\,, \tag{1.1}$$

where $\beta(r) = 1 - 2m/r$, $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta \cdot d\varphi^2$ is the standard metric of the unit 2-sphere in the Euclidean 3-space, $m = GM/c^2$, G is the universal gravitation constant, M is the mass of the body and c is the speed of the light.

Notice that

$$\beta(r) = 0 \qquad \text{for } r = 2m\,. \tag{1.2}$$

Since (1.1) is a solution of the Einstein equations in the empty space, it is physically meaningful to equip all $\{r > 2m\} \times \mathbf{R}$ with the metric (1.1), only if the radius r_M of the body is less that $2m$. In this case the Schwarzschild spacetime is an example of universe with a black hole. The name is justified by the fact that a light ray cannot leave the region $\{r \leq 2m\}$. If an astronaut «falls» in the black hole, he spends a finite «prop-

(*) Nella seduta del 10 novembre 1990.

er» time, but an observer far from the black hole does not see the astronaut to fall in it in a finite time.

The Schwarzschild spacetime is a physical example of a static Lorentz manifold.

DEFINITION (1.3). *Let* $(\mathscr{L}, \langle\,,\rangle_L)$ *be a Lorentz manifold.* $\mathscr{L}$ *is said to be static if: there exists a Riemann manifold* $(\mathscr{M}_0, \langle\,,\rangle_R)$ *of class* C^2 *such that* $(\mathscr{L}, \langle\,,\rangle_L)$ *is isometric to* $\mathscr{M}_0 \times \mathbf{R}$*, and, in the coordinate system* (x,t) $(x \in \mathscr{M}_0, t \in \mathbf{R})$,

$$\langle\,,\rangle_L = \langle\,,\rangle_R - \beta(x)\,dt^2 \tag{1.4}$$

where $\beta \in C^2(\mathscr{M}_0, \mathbf{R}^+ \setminus \{0\})$.

In order to look for geodesics on a Lorentz manifold we shall take advantage of the static structure.

Indeed the geodesics joining two given events $z_0 = (x_0, t_0)$ and $z_1 = (x_1, t_1)$ in $\mathscr{L}$ are the critical points of the functional

$$f(\gamma) = \int_0^1 \langle\dot\gamma, \dot\gamma\rangle_L\,ds = \int_0^1 \left(\langle\dot x(s), \dot x(s)\rangle_R - \beta(x(s))\,(\dot t(s))^2\right) ds \tag{1.5}$$

on the space of the smooth curves $\gamma(s) = (x(s), t(s))$ on $\mathscr{L}$ such that $\gamma(0) = z_0$, $\gamma(1) = z_1$. Here $\dot\gamma(s)$, $\dot x(s)$ and $\dot t(s)$ denote the derivatives of $\gamma(s)$, $x(s)$ and $t(s)$ respectively. The coordinate function t is called universal time and its existence means that there is way to synchronize all the watches in $\mathscr{M}_0$. The parameter s is proportional to the proper time which is the time measured by an observer moving along a geodesic.

We recall that if γ is a geodesic in $\mathscr{L}$ there exists a constant $E_\gamma \in \mathbf{R}$ such that, for every s, $E_\gamma = \langle\dot\gamma(s), \dot\gamma(s)\rangle_L$.

A geodesic γ is called *space-like, null* or *time-like* if E_γ is greater, equal or less then zero, respectively.

A time-like geodesic is physically interpreted as the world line of a material particle under the action of a gravitational field, while a null geodesic is the world line of a light ray. Space-like geodesics have less physical relevance, however they are useful to the study of the geometrical properties of a Lorentz manifold.

The functional (1.5) is indefinite, *i.e.* $\sup f = +\infty$ and $\inf f = -\infty$. This fact creates technical difficulties for the research of critical points of f. However in the static case we can reduce our problem to the study of a functional bounded from below if, for instance, the function β is bounded from above.

Indeed, as proved in [4] by a simple calculation, $\gamma(s) = (x(s), t(s))$ is a critical point for the functional f with $\gamma(0) = z_0$ and $\gamma(1) = z_1$, if and only if $x(s)$ is a critical point for the functional

$$J(x) = \int_0^1 \left(\langle\dot x(s), \dot x(s)\rangle_R - (t_1 - t_0)^2 \left[\int_0^1 \frac{1}{\beta(x(s))}\,ds \right]^{-1} \right) ds \tag{1.6}$$

with $x(0)=x_0$ and $x(1)=x_1$, and $t(s)$ solves the Cauchy problem

$$(1.7)\qquad \begin{cases} \dot t(s)=(t_1-t_0)\left[\int_0^1 \frac{1}{\beta(x(\tau))}\,d\tau\right]^{-1}\cdot\frac{1}{\beta(x(s))}\,. \\ t(0)=t_0\,. \end{cases}$$

Moreover for the critical points γ and x we have

$$(1.8)\qquad f(\gamma)=J(x)\,.$$

In the Theorems stated in sections 2 and 3 (whose proofs are in [5]), the function β is assumed to be bounded from above, so the functional J is bounded from below. Then, using suitable penalization methods of the functional J (which are necessary because of the lack of completeness of $(\mathcal{M}_0, \langle\,,\,\rangle_R)$) we get existence results for geodesics by a minimum argument and multiplicity results by the classical Lusternik and Schnirelmann theory (see *e.g.* [10]).

2. Static universes

Motivated by the previous considerations on the Schwarzschild spacetime we give the following

Definition (2.1). *Let U be an open, connected subset of a manifold $\mathcal{M}$ and let ∂U be its topological boundary. U is said to be* static universe (with a black hole) *if*

(i) $U=\mathcal{M}_0\times\mathbf{R}$ *is a static Lorentz manifold* (see (1.3));

(ii) $\sup_{\mathcal{M}_0}\beta<+\infty$, *where β is the function in* (1.4);

(iii) $\lim_{(x,t)\to z\in\partial U}\beta(x)=0$;

(iv) *for every $\delta>0$ the set $\{x\in\mathcal{M}_0:\ \beta(x)\geqslant\delta\}$ is complete (with respect to the Riemann structure of $\mathcal{M}_0$)*;

(v) *for every time-like geodesic $\gamma(s)=(x(s),\,t(s))$ in U such that* $\liminf_{s\to s_0^-}\beta(x(s))=0$, *we have* $\limsup_{s\to s_0^-}|t(s)|=+\infty$.

Remark (2.2). *Condition* (v) *says that if a material particle reaches the topological boundary of U, an observer far from the boundary does not see this event in a finite time, since its proper time is a reparametrization of the universal time. This condition justifies the name of the structure defined in* (2.1).

Notice that in general (v) *does not follow from* (iii). *In fact consider the Lorentz manifold*

$$(2.3)\qquad \{x\in\mathbf{R}:\ x>1\}\times\mathbf{R}\ \textit{with the metric}\ ds^2=dx^2-\beta(x)\,dt^2\,,$$
$$\textit{where }\beta\textit{ is bounded and }\beta(x)=x-1\textit{ if }x\leqslant 2;$$

A straightforward calculation shows that (2.3) *does not satisfy* (v).

By a symple calculation (see the appendix of [5]) we see that the spacetime $U = \{(r, \vartheta, \varphi): r > 2m\} \times \mathbf{R}$ *with the metric* (1.1) satisfies (v) of Definition (2.1). Then, clearly, it is a static universe.

The same computations show that, when $m^2 > e^2$, (m represents the gravitational mass and e the electric charge of the body), also the Reissner-Nordström spacetime is a static universe for $r > m + \sqrt{m^2 - e^2}$.

Assume that U is a static universe (with a black hole). We have the following results about the existence of a time-like geodesic joining two given events.

THEOREM (2.4). *Let* $z_0 = (x_0, t_0)$ *and* $z_1 = (x_1, t_1)$ *be events in* U. *There exists a time-like geodesic* γ *in* U *such that* $\gamma(0) = z_0$ *and* $\gamma(1) = z_1$ *if and only if* $\exists x \in C^1([0,1], \mathcal{M}_0)$: $x(0) = x_0$, $x(1) = x_1$ *and*

$$\left[\int_0^1 \frac{1}{\beta(x(s))}\, ds\right] \cdot \left[\int_0^1 \langle \dot{x}(s), \dot{x}(s) \rangle_R \, ds\right] < (t_1 - t_0)^2 . \tag{2.5}$$

REMARK. *Let us fix* x_0 *and* x_1. *Condition* (2.5) *is certainly satisfied if* $|t_1 - t_0|$ *is large enough, while it does not hold whenever* $|t_1 - t_0|$ *is small.*

Now let $\mathcal{L} \cong \mathcal{M}_0 \times \mathbf{R}$ be a static Lorentz manifold and (x_0, t_0), (x_1, t_1) two events in $\mathcal{L}$. If $(x(s), t(s))$ is a geodesic joining (x_0, t_0) and (x_1, t_1), since the metric tensor is independent of t, $(x(s), t(s) + \tau)$ is a geodesic joining $(x_0, t_0 + \tau)$ and $(x_1, t_1 + \tau)$. Then the number of geodesics in $\mathcal{L}$ joining two events (x_0, t_0) and (x_1, t_1) depends only on x_0, x_1 and $|t_1 - t_0|$.

We denote by $N(x_0, x_1, |t_1 - t_0|)$ the number of the time-like geodesics in U joining (x_0, t_0) and (x_1, t_1).

If U has a non trivial topology we get the following multiplicity result of geodesics joining z_0 and z_1.

THEOREM (2.6). *Let* $U \cong \mathcal{M}_0 \times \mathbf{R}$ *be a static universe* (*with a black hole*) *and* $\mathcal{M}_0$ *a Riemann manifold of class* C^3. *Assume that* $\mathcal{M}_0$ *is not contractible in itself. Then*

$$\lim_{|t_1 - t_0| \to +\infty} N(x_0, x_1, |t_1 - t_0|) = +\infty .$$

REMARK (2.7). *Condition* (ii) *of Definition* (2.1) *is essential to obtain our existence results. The Anti-de Sitter spacetime* (*see* e.g. [6, 9]) *furnishes a counterexample.*

However if $\beta(x)$ *goes to* $+\infty$ *with a mild rate, Theorems* (2.4) *and* (2.6) *still hold.*

Theorems (2.4) and (2.6) are related to some results of [1, 11, 12]. In these papers it is always assumed that the Lorentz manifolds are globally hyperbolic (but not necessarily static). However the assumption of global hyperbolicity is not always easy to verify.

3. Lorentzian manifolds with convex boundary

Now we consider the problem of the geodesical connectivity for Lorentz manifolds.

A Lorentz manifold $\mathcal{L}$ is said to be geodesically connected if for every z_0, z_1 in $\mathcal{L}$ there exists a geodesic γ in $\mathcal{L}$ such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$.

Clearly for studying the geodesical connectivity, it is necessary to consider also space like geodesics which are more difficult to deal with. The geodesical connectivity has not been treated in the previous works on this topic which deal only with time-like and light-like geodesics.

This problem has been faced for the first time in [2, 3] for stationary Lorentz manifolds without boundary. Here we consider the case of static Lorentz manifolds with singular boundary, in order to cover the case of the Schwarzschild spacetime. Indeed using the coordinates introduced by Kruskal in 1960 (see [6, 7]), to construct the maximal analytical extension of the Schwarzschild spacetime, we see that its topological boundary is not smooth.

For the study of the geodesical connectivity the condition of static universe is not appropriated. Indeed consider the Lorentz manifold

(3.1) $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 > 1\} \times \mathbf{R}$ *with metric* $ds^2 = dx^2 + dy^2 - \beta(x,y)\,dt^2$, *where* β *is bounded and* $\beta(x,y) = \left(\sqrt{x^2+y^2} - 1\right)^2$ *if* $\sqrt{x^2+y^2} \leq 2$.

Simple calculations show that (3.1) is a static universe while it is not geodesically connected (the events of the type (x_1, x_2, t_0) and $(-x_1, -x_2, t_0)$ cannot be joined by geodesics lying in the Lorentz manifold (3.1)).

For this reason we introduce a condition of convexity for the topological boundary. Denote by *grad* $\phi(z)$ the gradient of the function ϕ with respect to the Lorentz structure, *i.e.* the unique vector field G on $T_Z(\mathcal{L})$ (the tangent space of $\mathcal{L}$ at z) such that $\langle G, v\rangle_L = d\phi(z)\,v$ for all $v \in T_Z(\mathcal{L})$. Moreover denote by $H_L^{\phi}(z)\,[v,v]$ the Hessian of the function ϕ at z in the direction v, *i.e.* $(d^2/ds^2)(\phi(\gamma(s)))_{|s=0}$ where γ is a geodesic such that $\gamma(0) = z$ and $\dot\gamma(0) = v$.

Definition (3.2). *Let $\mathcal{L}$ be an open connected subset of a manifold $\mathcal{M}$ and $\partial\mathcal{L}$ its topological boundary. $\mathcal{L}$ is said to be* static Lorentz manifold with convex boundary *if*

(i) *$\mathcal{L} = \mathcal{M}_0 \times \mathbf{R}$ is a static Lorentz manifold (see (1.3));*

(ii) $\sup_{\mathcal{M}_0} \beta < +\infty$, *where β is the function in (1.4);*

(iii) *there exists $\phi \in C^2(\mathcal{L}, \mathbf{R}^+ \setminus \{0\})$ such that $\lim_{(x,t)\to z \in \partial\mathcal{L}} \phi(x,t) = 0$ and $\phi(x,t) = \phi(x,0) \equiv \phi(x)$ $\forall (x,t) \in \mathcal{L}$;*

(iv) *for every $\delta > 0$ the set $\{x \in \mathcal{M}_0 : \varphi(x) \geq \delta\}$ is complete (with respect to the Riemann structure of $\mathcal{M}_0$);*

(v) *there exists a neighbourhood N of $\partial\mathcal{L}$ and there exist ν, N, $M \in \mathbf{R}^+ \setminus \{0\}$*

such that in $N \cap \mathscr{L}$ *the function* ϕ *of* (iii) *satisfies:*

$$N \geq \langle \operatorname{grad} \phi(z), \operatorname{grad} \phi(z) \rangle_L \geq \nu, \tag{3.3}$$

$$H_L^{\phi}(z)[v,v] \leq M \cdot |\langle v,v \rangle_L| \cdot \phi(z) \qquad \forall v \in T_Z(\mathscr{L}). \tag{3.4}$$

REMARK. *The Schwarzschild spacetime satisfies* (v) *of* (3.2) *with the function* $\phi(r, \vartheta, \varphi, t) = \sqrt{1 - 2m/r}$ *(see* [6]*), so it is a static Lorentz manifold with convex boundary.*

Also the Reissner-Nordström spacetime $\left(\text{for } r > m + \sqrt{m^2 - e^2}\right)$ *is a static Lorentz manifold with convex boundary provided that* $m^2 > 9e^2/5$, *as we can verify* (*see* [6]) *using the function* $\phi(r, \vartheta, \varphi, t) = \sqrt{1 - 2m/r + e^2/r^2}$.

THEOREM (3.5). *A static Lorentz manifold with convex boundary is geodesically connected.*

REMARK (3.6). *In order to interpret assumption* (3.4), *notice that* ϕ *becomes zero on* $\partial\mathscr{L}$, *so* (3.3) *implies*

$$\limsup_{z \to z_0 \in \partial\mathscr{L}} H_L^{\phi}(z)[v,v] \leq 0 \qquad \text{for all } v \text{ such that } |\langle v,v \rangle_L| \leq 1. \tag{ϕ}$$

It can be proved that (ϕ) *is sufficient to guarantee the geodesical connectivity of* $\mathscr{L} \cup \partial\mathscr{L}$ *but not of* $\mathscr{L}$ *because* $\partial\mathscr{L}$ *is not smooth.*

In order to get the geodesical connectivity of $\mathscr{L}$ *it seems we need a control of the rate for which the limit in* (ϕ) *is achieved.*

The assumption (3.4) *provides this control.*

When the topology of $\mathscr{L}$ is not trivial we have the following multiplicity result about the space-like geodesics.

THEOREM (3.7). *Let* $\mathscr{L} \cong \mathscr{M}_0 \times \mathbf{R}$ *be a static Lorentz manifold with convex boundary and* $\mathscr{M}_0$ *a Riemann manifold of class* C^3. *Assume that* $\mathscr{M}_0$ *is not contractible in itself.*

Then for every $z_0, z_1 \in \mathscr{L}$ *there exists a sequence* $\{\gamma_n\}_{n \in \mathbf{N}}$ *of geodesics in* $\mathscr{L}$ *joining* z_0 *and* z_1 *such that*

$$\lim_{n \to +\infty} E_{\gamma_n} = +\infty .$$

REMARK (3.8). *Theorems* (2.4) *and* (2.6) *hold even for a static Lorentz manifold with convex boundary, while Theorems* (3.5) *and* (3.7) *in general do not hold for a static universe, as we can see using the Lorentz manifold* (3.1).

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REFERENCES

[1] A. AVEZ, *Essais de geometrie Riemanniene hyperbolique: applications to the relativité generale.* Ann. Inst. Fourier, 132, 1963, 105-90.

[2] V. BENCI - D. FORTUNATO, *Existence of geodesics for the Lorentz metric of a stationary gravitational*

field. Ann. Inst. H. Poincaré, Analyse non Lineaire, 7, 1990, 27-35.
[3] V. Benci - D. Fortunato, *On the existence of infinitely many geodesics on space-time manifolds.* Adv. Math., to appear.
[4] V. Benci - D. Fortunato - F. Giannoni, *On the existence of multiple geodesics in static space-times.* Ann. Inst. H. Poincaré, Analyse non Lineaire, to appear.
[5] V. Benci - D. Fortunato - F. Giannoni, *On the existence of geodesics in Lorentz manifolds with singular boundary.* Ist. Mat. Appl. Univ. Pisa, preprint.
[6] S. W. Hawking - G. F. Ellis, *The large scale structure of space-time.* Cambridge Univ. Press, 1973.
[7] M. D. Kruskal, *Maximal extension of Schwarzschild metric.* Phys. Rev., 119, 1960, 1743-1745.
[8] B. O'Neill, *Semi-Riemannian geometry with applications to relativity.* Academic Press Inc., New York-London 1983.
[9] R. Penrose, *Techniques of differential topology in relativity.* Conf. Board Math. Sci., 7, S.I.A.M. Philadelphia 1972.
[10] J. T. Schwartz, *Nonlinear functional analysis.* Gordon and Breach, New York 1969.
[11] H. J. Seifert, *Global connectivity by time-like geodesics.* Z. Natureforsch, 22*a*, 1970, 1356-60.
[12] K. Uhlenbeck, *A Morse theory for geodesics on a Lorentz manifold.* Topology, 14, 1975, 69-90.

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