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Γ-convergence of discrete approximations to interfaces with prescribed mean curvature

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Calcolo delle variazioni. — *Γ-convergence of discrete approximations to interfaces with prescribed mean curvature.* Nota di GIOVANNI BELLETTINI, MAURIZIO PAOLINI e CLAUDIO VERDI, presentata (*) dal Socio E. DE GIORGI.

ABSTRACT. — The numerical approximation of the minimum problem: $\min_{A \subseteq \Omega} \widetilde{\mathcal{F}}(A)$, is considered, where $\widetilde{\mathcal{F}}(A) = P_\Omega(A) + \cos(\theta)\mathcal{H}^{n-1}(\partial A \cap \partial\Omega) - \int_A \varkappa$. The solution to this problem is a set $A \subseteq \Omega \subset \boldsymbol{R}^n$ with prescribed mean curvature $\varkappa$ and contact angle θ at the intersection of ∂A with $\partial\Omega$. The functional $\widetilde{\mathcal{F}}$ is first relaxed with a sequence of nonconvex functionals defined in $H^1(\Omega)$ which, in turn, are discretized by finite elements. The Γ-convergence of the discrete functionals to $\widetilde{\mathcal{F}}$ as well as the compactness of any sequence of discrete absolute minimizers are proven.

KEY WORDS: Calculus of variations; Surfaces with prescribed mean curvature; Finite elements; Convergence of discrete approximations.

RIASSUNTO. — *Γ-convergenza di approssimazioni discrete di interfacce con curvatura media prescritta.* Si studia l'approssimazione numerica del seguente problema di minimo: $\min_{A \subseteq \Omega} \widetilde{\mathcal{F}}(A)$, ove $\widetilde{\mathcal{F}}(A) = P_\Omega(A) + \cos(\theta)\mathcal{H}^{n-1}(\partial A \cap \partial\Omega) - \int_A \varkappa$, teso alla ricerca di un insieme $A \subseteq \Omega \subset \boldsymbol{R}^n$ con curvatura media $\varkappa$ e angolo di contatto θ all'intersezione di ∂A con $\partial\Omega$. Il funzionale $\widetilde{\mathcal{F}}$ viene preliminarmente rilassato mediante una successione di funzionali non convessi definiti in $H^1(\Omega)$, che sono successivamente discretizzati con elementi finiti. Si dimostrano la Γ-convergenza dei funzionali discreti al funzionale $\widetilde{\mathcal{F}}$ e la compattezza di qualunque successione di minimi assoluti dei funzionali discreti.

0. INTRODUCTION

Recently, E. De Giorgi has drawn attention to the numerical approximation of problems in the calculus of variations that fall within the general setting given by him during the last few years [8-10]. The discretization of such problems is very difficult, because of the lack of convexity and regularity properties of the functionals involved. This paper addresses the questions formulated by De Giorgi for a particular functional relevant in a number of different contexts involving surface tension, such as fluid phase transition theory, capillarity theory, and image segmentation in computer vision theory [1, 2, 7, 14, 16].

We present here a numerical approximation to a functional whose minimizers give rise to interfaces with prescribed mean curvature. Such a functional is first regularized following an idea of L. Modica and S. Mortola [15], who proved convergence of the relaxed functionals; see also [3, 14, 17]. It is next discretized by means of piecewise linear finite elements with numerical quadrature [4], thus allowing the actual implementation on a computer. We demonstrate the convergence of the discrete minimizers to a solution of the continuous problem. Numerical experiments are in progress at the I.A.N. of C.N.R. in Pavia and implementation

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details on the numerical algorithm as well as several experiments will appear in [4, 5].

More specifically, given an open bounded convex set $\Omega \subset \mathbf{R}^n$ ($n \geq 2$) with piecewise C^2 boundary, a function $\varkappa \in L^\infty(\Omega)$, and $\theta \in \mathbf{R}$, we consider the minimum problem

$$(0.1)\quad \min_{A \subseteq \Omega} \tilde{\mathcal{F}}(A), \qquad \text{where} \qquad \tilde{\mathcal{F}}(A) := P_\Omega(A) + \cos(\theta)\mathcal{H}^{n-1}(\partial A \cap \partial\Omega) - \int_A \varkappa dx.$$

Here $P_\Omega(A)$ denotes the perimeter of A in Ω [18] and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure [12]. The solution to this problem is a measurable set $A \subseteq \Omega$ whose boundary has mean curvature $\varkappa$ and contact angle θ at the intersection of ∂A with $\partial\Omega$ [13].

In order to introduce a rigorous formulation of (0.1) we need further notations. Let $BV(\Omega; \{-1, 1\})$ be the space of the functions of bounded variation with values in $\{-1, 1\}$ and denote by S_v and $\mathrm{tr}(v)$ the *jump* set and the trace on $\partial\Omega$, respectively, of the BV function v [18]. Set $\mu := \cos(\theta)$ and define the following energy functional in $L^1(\Omega)$:

$$\mathcal{F}(v) := \begin{cases} 2\mathcal{H}^{n-1}(S_v) + \int_{\partial\Omega} |\mathrm{tr}(v) + \mu| d\mathcal{H}^{n-1}(x) - \int_\Omega \varkappa v dx, & \text{if } v \in D(\mathcal{F}), \\ +\infty, & \text{if } v \in L^1(\Omega) \setminus D(\mathcal{F}), \end{cases}$$

where $D(\mathcal{F}) := BV(\Omega; \{-1, 1\})$. It is well known [13] that $\mathcal{F}$ admits at least a minimizer u so that the set $A_u := \{x \in \Omega : u(x) = +1\}$ is a solution to problem (0.1). In addition, we have $\mathcal{F}(v) = 2\tilde{\mathcal{F}}(A_v) + (1 - \mu)\mathcal{H}^{n-1}(\partial\Omega) + \int_\Omega \varkappa$, for all $v \in D(\mathcal{F})$. As a generalization, from now on we will consider μ to be a piecewise constant function $\mu \in BV(\partial\Omega; [-1, 1])$.

We first approximate $\mathcal{F}$ with a family $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ of regular nonconvex functionals defined in $H^1(\Omega)$ which, in turn, will be discretized by finite elements: ε is the *relaxation* parameter and h is the *meshsize*. In order to define the relaxed functionals $\mathcal{F}_\varepsilon$ we need some preparations. Let $\omega : \mathbf{R} \to \mathbf{R}^+ \cup \{+\infty\}$ be defined by

$$(0.2)\qquad \omega(t) := \begin{cases} 1 - t^2 & \text{if } |t| \leq 1, \\ +\infty, & \text{if } |t| > 1; \end{cases}$$

different choices for ω are discussed in Remark 2.1. In addition, set

$$(0.3)\qquad \varphi(t) := \int_0^t \sqrt{\omega(s)}\, ds, \qquad \forall t \in [-1, 1],$$

$$(0.4)\qquad \delta(t_1, t_2) := \left| \int_{t_1}^{t_2} \sqrt{\omega(t)}\, dt \right| = |\varphi(t_2) - \varphi(t_1)|, \qquad \forall t_1, t_2 \in [-1, 1],$$

$$c_0 := \int_{-1}^{1} \sqrt{\omega(t)}dt = \delta(-1,1) = 2\varphi(1) = -2\varphi(-1) = \pi/2. \tag{0.5}$$

Since φ is an (odd) strictly increasing function, the following piecewise constant function $g \in BV(\partial\Omega; [-1,1])$ is well defined

$$g(x) := \varphi^{-1}(-2^{-1} c_0 \mu(x)), \qquad \forall x \in \partial\Omega. \tag{0.6}$$

Let $\{g_1, \dots, g_m\}$ be the values of g and set $\Gamma_i := \{x \in \partial\Omega : g(x) = g_i\}$, for all $1 \leqslant i \leqslant m$. Note that $\mathcal{H}^{n-2}(S_g) < +\infty$. For any $\varepsilon > 0$, let $g_\varepsilon \in C^{0,1}(\partial\Omega; [-1,1])$ approximate g in the sense that

$$g_\varepsilon(x) = g(x), \qquad \text{if } \operatorname{dist}(x, S_g) \geqslant \varepsilon, \qquad \operatorname{Lip}(g_\varepsilon) \leqslant L_g/\varepsilon, \tag{0.7}$$

where L_g is a constant independent of ε. It is obvious that $g_\varepsilon \to g$ in $L^1(\partial\Omega)$, as $\varepsilon \to 0$.

We are now in a position to define the relaxed functionals as

$$\mathcal{F}_\varepsilon(v) := \begin{cases} \displaystyle\int_\Omega [\varepsilon|\nabla v|^2 + \varepsilon^{-1}\omega(v) - c_0 \varkappa v]dx, & \text{if } v \in D(\mathcal{F}_\varepsilon), \\ +\infty, & \text{if } v \in L^1(\Omega) \setminus D(\mathcal{F}_\varepsilon), \end{cases}$$

where $D(\mathcal{F}_\varepsilon) := \{v \in H^1(\Omega; [-1,1]) : v = g_\varepsilon \text{ on } \partial\Omega\}$. The existence of a solution u_ε to the minimum problem for $\mathcal{F}_\varepsilon$ can be proven by direct methods. We have that $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} c_0 \mathcal{F}$, as $\varepsilon \to 0$ (we refer to [11] for basic issues about Γ-convergence). In fact, it is known [17] that

$$\mathcal{F}_\varepsilon \underset{\varepsilon\to 0}{\xrightarrow{\Gamma}} \hat{\mathcal{F}}, \quad \text{where} \quad \hat{\mathcal{F}}(v) := 2c_0 \mathcal{H}^{n-1}(S_v) + 2\int_{\partial\Omega} \delta(\operatorname{tr}(v)(x), g(x)) d\mathcal{H}^{n-1}(x) - c_0 \int_\Omega \varkappa v\, dx,$$

which, on using (0.4), (0.5), and (0.6), yields the asserted Γ-convergence result because

$$\delta(\operatorname{tr}(v), g) = |\varphi(\operatorname{tr}(v)) - \varphi(g)| = 2^{-1} c_0 |\operatorname{tr}(v) + \mu|, \qquad \forall v \in D(\mathcal{F}). \tag{0.8}$$

At this stage, we can introduce a discretization of the relaxed functional $\mathcal{F}_\varepsilon$ by piecewise linear finite elements. Let us denote by $\{\mathcal{S}_h\}_{h>0}$ a regular family of partitions of Ω into simplices [6, p. 132]. For the sake of simplicity we shall assume that Ω is a polyhedron, so that $\overline{\Omega} \equiv \Omega_h := \cup_{S \in \mathcal{S}_h} S$, for all $h > 0$. This, as well as the convexity of the domain, is just assumed to avoid technical difficulties. Set $\partial_a\Omega := \{x \in \partial\Omega : \partial\Omega \notin C^2 \text{ in } x\}$ and note that $\mathcal{H}^{n-2}(\partial_a\Omega) < +\infty$. Let $V_h \subset H^1(\Omega)$ indicate the usual piecewise linear finite element space over $\mathcal{S}_h$ and $\Pi_h : C^0(\overline{\Omega}) \to V_h$ the Lagrange interpolation operator.

Finally, for any $\varepsilon > 0$, let $\varkappa_\varepsilon \in C^{0,1}(\overline{\Omega})$ approximate the *curvature* function $\varkappa$ so that

$$\|\varkappa_\varepsilon\|_{L^\infty(\Omega)} \leqslant C, \qquad \operatorname{Lip}(\varkappa_\varepsilon) \leqslant L_\varkappa/\varepsilon, \qquad \varkappa_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \varkappa \text{ in } L^1(\Omega), \tag{0.9}$$

where C and $L_\varkappa$ are constants independent of ε.

We are now in a position to introduce the discrete functionals

$$(0.10) \qquad \mathcal{F}_{\varepsilon,h}(v) := \begin{cases} \displaystyle\int_\Omega [\varepsilon|\nabla v|^2 + \varepsilon^{-1}\Pi_h\,\omega(v) - c_0\Pi_h(\varkappa_\varepsilon v)]dx, & \text{if } v \in D(\mathcal{F}_{\varepsilon,h}), \\ +\infty, & \text{if } v \in L^1(\Omega)\setminus D(\mathcal{F}_{\varepsilon,h}), \end{cases}$$

where $D(\mathcal{F}_{\varepsilon,h}) := \{v \in V_h : |v| \leqslant 1 \text{ in } \Omega, v = \Pi_h g_\varepsilon \text{ on } \partial\Omega\}$. The existence of a solution $u_{\varepsilon,h}$ to the minimum problem for $\mathcal{F}_{\varepsilon,h}$ is trivial. The integrals in (0.10) can be evaluated via the vertex quadrature rule, which is exact for piecewise linear functions. On the other hand, the interpolation operator Π_h in (0.10) will introduce extra difficulties in proving the main Γ-convergence result in our paper, which read as follows:

THEOREM 0.1. *Let $h = o(\varepsilon)$. Then the sequence $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon,h}$ Γ-converges to $c_0\mathcal{F}$, as $\varepsilon \to 0$. Moreover, any family $\{u_{\varepsilon,h}\}_{\varepsilon,h}$ of absolute minimizers of $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon,h}$ is relatively compact in $L^1(\Omega)$, and any limit point u minimizes $\mathcal{F}$.*

The proof of this Theorem will be given in full detail in §2 whereas, in §1, we study the functional $\mathcal{F}_\varepsilon$ in a one dimensional domain with $\varkappa = 0$.

1. ONE DIMENSIONAL MINIMIZERS

Our present purpose is to calculate an absolute minimizer u_ε of the functional $\mathcal{F}_\varepsilon$ in the one dimensional domain $\Omega := (-l, l)$ with curvature $\varkappa = 0$ and boundary conditions $g(-l) = -1, g(l) = 1$, namely,

$$(1.1) \qquad \mathcal{G}_\varepsilon(v) := \int_{-l}^{l} [\varepsilon v'^2 + \varepsilon^{-1}\omega(v)]dx, \qquad \text{if } v \in W^{2,\infty}(\Omega): v(-l) = -1, v(l) = 1.$$

It is easy to check that any absolute minimizer of $\mathcal{G}_\varepsilon$ has to be a monotone function. Since the following inequality holds for any nondecreasing function v such that $v(-l) = -1$ and $v(l) = 1$ (use the Young inequality and (0.5)),

$$\mathcal{G}_\varepsilon(v) \geqslant 2\int_{-l}^{l} v'(x)\sqrt{\omega(v(x))}dx = 2\int_{-1}^{1}\sqrt{\omega(t)}dt = 2c_0 = \pi,$$

then any monotone function u_ε satisfying the imposed boundary conditions and the equality

$$(1.2) \qquad \varepsilon^2 u_\varepsilon'(x)^2 = \omega(u_\varepsilon(x)), \qquad \forall x \in \Omega,$$

is an absolute minimizer of $\mathcal{G}_\varepsilon$. In addition, the minimal energy is

$$(1.3) \qquad \mathcal{G}_\varepsilon(u_\varepsilon) = 2c_0 = \pi.$$

Thus, for all ε sufficiently small and a such that $(a - \varepsilon\pi/2, a + \varepsilon\pi/2) \subset \Omega$, we have

$$(1.4) \qquad u_\varepsilon(x) = \begin{cases} -1, & \text{if } x \in [-l, a - \varepsilon\pi/2], \\ \sin[\varepsilon^{-1}(x-a)], & \text{if } x \in (a - \varepsilon\pi/2, a + \varepsilon\pi/2), \\ 1, & \text{if } x \in [a + \varepsilon\pi/2, l]. \end{cases}$$

2. Proof of the main result

Our goal in this section is to demonstrate Theorem 0.1 by proving first the Γ-convergence of the functionals, and next the $L^1(\Omega)$-compactness of any sequence of minimizers.

Theorem 2.1. *If $h = o(\varepsilon)$, then $\Gamma - \lim_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,h} = c_0 \mathcal{F}$. More specifically, the two following properties hold:*

i) For any $v_0 \in L^1(\Omega)$ and any sequence $\{v_{\varepsilon,h}\}_{\varepsilon,h} \in L^1(\Omega)$ converging to v_0 in $L^1(\Omega)$, we have $c_0 \mathcal{F}(v_0) \leq \liminf_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,h}(v_{\varepsilon,h})$.

ii) For any $v_0 \in L^1(\Omega)$, there exists a sequence $\{v_{\varepsilon,h}\}_{\varepsilon,h} \in L^1(\Omega)$ converging to v_0 in $L^1(\Omega)$ such that $c_0 \mathcal{F}(v_0) \geq \limsup_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,h}(v_{\varepsilon,h})$.

Proof. Let us split the functionals $\mathcal{F}$ and $\mathcal{F}_{\varepsilon,h}$, with obvious notation, as follows:

$$\mathcal{F}(v) = \left[2\mathcal{H}^{n-1}(S_v) + \int_{\partial\Omega} |\mathrm{tr}(v) + \mu| d\mathcal{H}^{n-1} \right] - \int_{\Omega} \varkappa v =: \mathcal{F}^1(v) + \mathcal{F}^2(v), \qquad \forall v \in D(\mathcal{F}),$$

$$\mathcal{F}_{\varepsilon,h}(v) = \int_{\Omega} [\varepsilon |\nabla v|^2 + \varepsilon^{-1} \omega(v)] - c_0 \int_{\Omega} \Pi_h(\varkappa_\varepsilon v) + \int_{\Omega} \varepsilon^{-1} [\Pi_h \omega(v) - \omega(v)] =$$

$$=: \mathcal{F}^1_{\varepsilon,h}(v) + c_0 \mathcal{F}^2_{\varepsilon,h}(v) + \mathcal{F}^3_{\varepsilon,h}(v), \qquad \forall v \in D(\mathcal{F}_{\varepsilon,h}).$$

Proof of (i). Let $v_0 \in L^1(\Omega)$ and $v_{\varepsilon,h} \to v_0$ in $L^1(\Omega)$, as $\varepsilon \to 0$. We can suppose, possibly taking a subsequence, that $v_{\varepsilon,h} \in D(\mathcal{F}_{\varepsilon,h})$ and $\lim_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,h}(v_{\varepsilon,h}) < +\infty$, otherwise ($i$) is obvious. Then, by virtue of the properties $|v_{\varepsilon,h}| \leq 1$ and (0.9), we have that $\left\{ \int_\Omega \Pi_h(\varkappa_\varepsilon v_{\varepsilon,h}) dx \right\}_{\varepsilon,h}$ is equibounded with respect to ε; hence, both $\left\{ \int_\Omega \varepsilon^{-1} \Pi_h \omega(v_{\varepsilon,h}) dx \right\}_{\varepsilon,h}$ and $\left\{ \int_\Omega \varepsilon |\nabla v_{\varepsilon,h}|^2 dx \right\}_{\varepsilon,h}$ are equibounded. In particular, the latter property reads as

$$\|\nabla v_{\varepsilon,h}\|_{L^2(\Omega)} \leq C\varepsilon^{-1/2}. \tag{2.1}$$

Now we deal with each functional $\mathcal{F}^1_{\varepsilon,h}$, $\mathcal{F}^2_{\varepsilon,h}$, and $\mathcal{F}^3_{\varepsilon,h}$, separately.

Step 1: *Behaviour of $\mathcal{F}^3_{\varepsilon,h}$*. Using well known properties of the interpolation operator Π_h and (2.1), we get

$$|\mathcal{F}^3_{\varepsilon,h}(v_{\varepsilon,h})| \leq \varepsilon^{-1} \sum_{S \in S_h} \int_S |\Pi_h \omega(v_{\varepsilon,h}) - \omega(v_{\varepsilon,h})| dx \leq Ch^2 \varepsilon^{-1} \sum_{S \in S_h} \|D^2 \omega(v_{\varepsilon,h})\|_{L^1(S)} \leq$$

$$\leq Ch^2 \varepsilon^{-1} \mathrm{Lip}(\omega') \sum_{S \in S_h} \|\nabla v_{\varepsilon,h}\|^2_{L^2(S)} = Ch^2 \varepsilon^{-1} \mathrm{Lip}(\omega') \|\nabla v_{\varepsilon,h}\|^2_{L^2(\Omega)} \leq Ch^2/\varepsilon^2.$$

Thus, enforcing the relation $h = o(\varepsilon)$, we obtain

$$\lim_{\varepsilon \to 0} \mathcal{F}^3_{\varepsilon,h}(v_{\varepsilon,h}) = 0. \tag{2.2}$$

Step 2: *Behaviour of $\mathcal{F}^1_{\varepsilon,h}$*. We closely follow [17, Lemma 1]. *Step* 1 guarantees that both $\{\mathcal{F}^1_{\varepsilon,h}(v_{\varepsilon,h})\}_{\varepsilon,h}$ and $\left\{\int_\Omega \varepsilon^{-1}\omega(v_{\varepsilon,h})dx\right\}_{\varepsilon,h}$ are equibounded with respect to ε. In particular, the latter property implies $|v_0|=1$ *a.e.* in Ω. Note that *step* 2 is actually independent of *step* 1 because, at this stage, the equiboundedness of $\{\mathcal{F}^1_{\varepsilon,h}(v_{\varepsilon,h})\}_{\varepsilon,h}$ could be assumed; otherwise the assertion (2.3) is obvious. As we shall see, here both parameters ε and h could go to 0 independently. Since Ω is a convex set, for any $x\in \boldsymbol{R}^n\setminus\Omega$, there exists a unique point $r(x)\in\partial\Omega$ so that $\mathrm{dist}(x,r(x))=\mathrm{dist}(x,\partial\Omega)$. For any $\sigma>0$, let $E_\sigma:=\{x\in\boldsymbol{R}^n\setminus\Omega\colon \mathrm{dist}(x,\partial\Omega)<\sigma\}$ and $S_\sigma:=\{x\in E_\sigma: r(x)\in\partial_a\Omega\cap S_g\}$. We extend the functions v_0 and $v_{\varepsilon,h}$ on $\Omega_\sigma:=\Omega\cup E_\sigma$ as follows:

$$\hat v_0(x):=\begin{cases} v_0(x), & \text{if } x\in\Omega,\\ g(r(x)), & \text{if } x\in E_\sigma\setminus S_\sigma,\\ g_\varepsilon(r(x)), & \text{if } x\in S_\sigma,\end{cases}\qquad \hat v_{\varepsilon,h}(x):=\begin{cases} v_{\varepsilon,h}(x), & \text{if } x\in\Omega,\\ (\Pi_h g_\varepsilon)(r(x)), & \text{if } x\in E_\sigma.\end{cases}$$

Note that, for all $x\in S_g$, $g_\varepsilon(x)$ can be supposed independent of ε; hence $\hat v_0$ does not depend on ε. Since $\mathcal{H}^{n-2}(S_g\cup\partial_a\Omega)<+\infty$, it is easy to check that $\hat v_0\in L^1(\Omega_\sigma;[-1,1])$ and $\hat v_{\varepsilon,h}\in H^1(\Omega_\sigma;[-1,1])$. In addition, using (0.7), we have that $\hat v_{\varepsilon,h}\to\hat v_0$ in $L^1(\Omega_\sigma)$, as $\varepsilon\to0$, because $\hat v_0(x)=\hat v_{\varepsilon,h}(x)$, for all $x\in E_\sigma\setminus E_{\sigma,\varepsilon}$, where $E_{\sigma,\varepsilon}:=\{x\in E_\sigma:\mathrm{dist}(r(x),S_g)<\varepsilon\}\setminus S_\sigma$, and $\mathcal{H}^n(E_{\sigma,\varepsilon})=O(\varepsilon\sigma)$. Now, since $v_{\varepsilon,h}\in D(\mathcal{F}_{\varepsilon,h})$, using the Young inequality and definition (0.3), we get

$$\frac{1}{2}\mathcal{F}^1_{\varepsilon,h}(v_{\varepsilon,h})\geqslant\int_\Omega|\nabla v_{\varepsilon,h}|\sqrt{\omega(v_{\varepsilon,h})}dx=\int_{\Omega_\sigma}|\nabla\varphi(\hat v_{\varepsilon,h})|dx-\int_{E_\sigma}|\nabla\varphi(\hat v_{\varepsilon,h})|dx=:I^2+II^2.$$

Since $\hat v_{\varepsilon,h}$ is locally constant on $E_\sigma\setminus E_{\sigma,\varepsilon}$, using (0.7) we get $|II^2|\leqslant C\sigma$. For any $v\in BV(\Omega)$, let $\int_\Omega|Dv|$ denote the total variation of the Radon measure Dv on Ω. Since $\varphi(\hat v_{\varepsilon,h})\to\varphi(\hat v_0)$ in $L^1(\Omega_\sigma)$, as $\varepsilon\to0$, the equiboundedness of $\{\mathcal{F}^1_{\varepsilon,h}(v_{\varepsilon,h})\}_{\varepsilon,h}$ and the semicontinuity of the total variation [18] give $\varphi(\hat v_0)\in BV(\Omega_\sigma)$ (hence $v_0\in D(\mathcal{F})$) and $\liminf_{\varepsilon\to0} I^2\geqslant\int_{\Omega_\sigma}|D\varphi(\hat v_0)|$, respectively. Now, (0.5) yields $\varphi(v_0)=2^{-1}c_0v_0$; hence, using (0.8) we have

$$\int_{\Omega_\sigma}|D\varphi(\hat v_0)|=\frac{c_0}{2}\int_\Omega|Dv_0|+\int_{\Omega_\sigma\setminus\bar\Omega}|D\varphi(\hat v_0)|+\int_{\partial\Omega}|\varphi(\mathrm{tr}(v_0))-\varphi(g)|d\mathcal{H}^{n-1}\geqslant\frac{c_0}{2}\mathcal{F}^1(v_0).$$

Thus, letting $\sigma\to0$, we conclude that

$$c_0\mathcal{F}^1(v_0)\leqslant\liminf_{\varepsilon\to0}\mathcal{F}^1_{\varepsilon,h}(v_{\varepsilon,h}).\tag{2.3}$$

Step 3: *Behaviour of* $\mathcal{F}^2_{\varepsilon,h}$. We first split $\mathcal{F}^2_{\varepsilon,h}(v_{\varepsilon,h})-\mathcal{F}^2(v_0)$ as follows:

$$\mathcal{F}^2_{\varepsilon,h}(v_{\varepsilon,h})-\mathcal{F}^2(v_0)=\int_\Omega[\varkappa_\varepsilon v_{\varepsilon,h}-\Pi_h(\varkappa_\varepsilon v_{\varepsilon,h})]dx+\int_\Omega[\varkappa v_0-\varkappa_\varepsilon v_{\varepsilon,h}]dx=:I^3+II^3.$$

Then, using well known properties of the interpolation operator Π_h, $|v_{\varepsilon,h}|\leqslant1$, (0.9), and (2.1), we can estimate I^3 as follows:

$$|I^3| \leq Ch[\|\nabla v_{\varepsilon,h}\|_{L^1(\Omega)} + \|\nabla \varkappa_\varepsilon\|_{L^1(\Omega)}] \leq Ch/\varepsilon.$$

On the other hand, we have $\lim_{\varepsilon \to 0} II^3 = 0$. Thus, imposing the relation $h = o(\varepsilon)$, we conclude that

$$\lim_{\varepsilon \to 0} \mathcal{F}^2_{\varepsilon,h}(v_{\varepsilon,h}) = \mathcal{F}^2(v_0). \tag{2.4}$$

The assertion (*i*) is thus proven in view of (2.2)-(2.4).

Proof of (*ii*). Let $v_0 \in L^1(\Omega)$. We can suppose $v_0 \in D(\mathcal{F})$, otherwise (*ii*) is obvious. Hence, the approximating sequence $\{v_{\varepsilon,h}\}_{\varepsilon,h}$ has to satisfy $v_{\varepsilon,h} \in D(\mathcal{F}_{\varepsilon,h})$. First we construct $v_\varepsilon \in D(\mathcal{F}_\varepsilon)$, closely following the ideas of [14, Prop. 2]. We need some preparations. Since $v_0 \in D(\mathcal{F}) = BV(\Omega; \{-1, 1\})$, it is well known that v_0 is the characteristic function of a set A of finite perimeter in Ω, namely, for *a.e.* $x \in \Omega$,

$$v_0(x) = \chi_A(x) := \begin{cases} +1, & \forall x \in A, \\ -1, & \forall x \in \Omega \setminus A. \end{cases} \tag{2.5}$$

Set $\partial_a A := \{x \in \partial A \cap \Omega : \partial A \notin C^2 \text{ in } x\}$. Using a diagonal argument ([2, Appendix]), we can suppose that A is a polyhedron; hence $\mathcal{H}^{n-2}(\partial_a A) < +\infty$. Let β be the minimum angle between the faces of $\partial\Omega$ and those of $\partial A \cap \Omega$, and $\eta := \cot(\beta/2)$. Employing the symbol Q to denote either A or Ω and symbol $\partial\Omega$ for $\partial A \cap \Omega$ or $\partial\Omega$, let us introduce the following notations:

$$d_Q(x) := \begin{cases} \operatorname{dist}(x, \partial Q), & \text{if } x \in Q, \\ -\operatorname{dist}(x, \partial Q), & \text{if } x \in \Omega \setminus Q; \end{cases}$$

$$\begin{aligned}
&\boldsymbol{r}_Q(x) := \{y \in \partial Q : \operatorname{dist}(y, x) = d_Q(x)\}, \qquad a.e.\ x \in \Omega; \\
&Q_t := \{x \in \overline{\Omega} : d_Q(x) = t\}, \qquad \forall t \in \boldsymbol{R}; \\
&L^A_\varepsilon := \{x \in \Omega : |d_A(x)| \leq 2^{-1}\pi\varepsilon\}; \\
&T^A_\varepsilon := \{x \in L^A_\varepsilon : \operatorname{dist}(\boldsymbol{r}_A(x), \partial_a A) \leq 2^{-1}\pi\eta\varepsilon\}; \\
&L^\Omega_\varepsilon := \{x \in \Omega : d_\Omega(x) \leq \pi\varepsilon\}; \qquad T^\Omega_\varepsilon := \{x \in L^\Omega_\varepsilon : \operatorname{dist}(\boldsymbol{r}_\Omega(x), \partial_a \Omega) \leq \pi\eta\varepsilon\}; \\
&L^{\Gamma_i}_\varepsilon := \{x \in L^\Omega_\varepsilon \setminus T^\Omega_\varepsilon : r_\Omega(x) \in \Gamma_i\}, \qquad \forall 1 \leq i \leq m; \\
&T^g_\varepsilon := \{x \in L^\Omega_\varepsilon \setminus T^\Omega_\varepsilon : \operatorname{dist}(r_\Omega(x), S_g) \leq \varepsilon\}; \\
&T^c_\varepsilon := L^A_\varepsilon \cap L^\Omega_\varepsilon; \qquad L_\varepsilon := L^A_\varepsilon \cup L^\Omega_\varepsilon; \qquad T_\varepsilon := T^g_\varepsilon \cup T^\Omega_\varepsilon \cup T^c_\varepsilon \cup T^A_\varepsilon; \\
&L_{\varepsilon,h} := \cup\{S \in \mathcal{S}_h : S \cap L_\varepsilon \neq \emptyset\}; \qquad T_{\varepsilon,h} := \cup\{S \in \mathcal{S}_h : S \cap T_\varepsilon \neq \emptyset\}.
\end{aligned}$$

Note that, for all $x \in L^\Omega_\varepsilon \setminus T^\Omega_\varepsilon$, $\boldsymbol{r}_\Omega(x)$ is a single valued function, that is $\boldsymbol{r}_\Omega(x) =: \{r_\Omega(x)\}$. In addition, it is easy to check that

$$|\nabla d_Q(x)| = 1, \qquad a.e.\ x \in \Omega, \qquad \lim_{t \to 0} \mathcal{H}^{n-1}(Q_t) = \mathcal{H}^{n-1}(\partial Q), \tag{2.6}$$

$$\mathcal{H}^n(\mathcal{L}_\varepsilon),\ \mathcal{H}^n(L_{\varepsilon,h}) = O(\varepsilon), \qquad \mathcal{H}^n(T_\varepsilon),\ \mathcal{H}^n(T_{\varepsilon,h}) = O(\varepsilon^2). \tag{2.7}$$

Now we introduce the following function $\gamma: \mathbf{R} \to [-1, 1]$:

$$\gamma(t) := \begin{cases} -1, & \text{if } t < -\pi/2, \\ \sin(t), & \text{if } t \in [-\pi/2, \pi/2], \\ +1, & \text{if } t > \pi/2, \end{cases} \tag{2.8}$$

and define $\gamma_\varepsilon(t) := \gamma(t/\varepsilon)$. As observed in (1.4), γ_ε is an absolute minimizer of the one dimensional functional $\mathcal{G}_\varepsilon$ defined in (1.1). Obviously we have

$$|\gamma_\varepsilon'(t)| \leq \varepsilon^{-1}, \qquad |\gamma_\varepsilon''(t)| \leq \varepsilon^{-2}, \qquad a.e.\ t \in \mathbf{R}. \tag{2.9}$$

Using γ_ε we can construct the approximating sequence $\{v_{\varepsilon,h}\}_{\varepsilon,h}$. First, we define a function v_ε on $\Omega \setminus T_\varepsilon$ as follows:

$$v_\varepsilon(x) := \begin{cases} v_0(x), & \text{if } x \in \Omega \setminus L_\varepsilon, \\ \gamma_\varepsilon(d_A(x)), & \text{if } x \in L_\varepsilon^A \setminus T_\varepsilon, \\ \gamma_\varepsilon(\varepsilon \arcsin(g_i) + d_\Omega(x)), & \text{if } x \in (L_\varepsilon^{\Gamma_i} \cap A) \setminus T_\varepsilon, 1 \leq i \leq m, \\ \gamma_\varepsilon(\varepsilon \arcsin(g_i) - d_\Omega(x)), & \text{if } x \in (L_\varepsilon^{\Gamma_i} \setminus A) \setminus T_\varepsilon, 1 \leq i \leq m. \end{cases} \tag{2.10}$$

Because of the particular shape of each connected component of T_ε, using a standard extension theorem, [12, Theorem 2.10.43], v_ε can be extended on the whole Ω as a Lipschitz continuous function (with Lipschitz constant C/ε) so that $v_\varepsilon \in D(\mathcal{F}_\varepsilon)$. In addition, $v_\varepsilon \to v_0$ in $L^1(\Omega)$, as $\varepsilon \to 0$, because

$$\int_\Omega |v_\varepsilon - v_0| dx = \int_{L_\varepsilon} |v_\varepsilon - v_0| dx \leq 2\mathcal{H}^n(L_\varepsilon) = O(\varepsilon)$$

(use (2.7)). Then, we define $\{v_{\varepsilon,h}\}_{\varepsilon,h}$ as follows:

$$v_{\varepsilon,h} := \Pi_h v_\varepsilon \in V_h.$$

Obviously we have $v_{\varepsilon,h} \in D(\mathcal{F}_{\varepsilon,h})$ and

$$|\nabla v_{\varepsilon,h}| \leq |\nabla v_\varepsilon| \leq C/\varepsilon, \qquad a.e.\ \text{in } \Omega. \tag{2.11}$$

In addition, by virtue of the assumption $h = o(\varepsilon)$, we have $v_{\varepsilon,h} \to v_0$ in $L^1(\Omega)$, as $\varepsilon \to 0$.

Step 4: *Behaviour of* $\mathcal{F}^1_{\varepsilon,h}$. Note that, by virtue of (2.5) and (0.8), $\mathcal{F}^1(v_0)$ can be represented as follows:

$$\mathcal{F}^1(v_0) = 2\mathcal{H}^{n-1}(\partial A \cap \Omega) + \\ + \frac{2}{c_0} \sum_{i=1}^m [\delta(-1, g_i)\mathcal{H}^{n-1}(\Gamma_i \setminus \partial A) + \delta(1, g_i)\mathcal{H}^{n-1}(\Gamma_i \cap \partial A)]. \tag{2.12}$$

Since $v_{\varepsilon,h} \in D(\mathcal{F}_{\varepsilon,h})$, we can split $\mathcal{F}^1_{\varepsilon,h}(v_{\varepsilon,h})$ as follows:

$$\mathcal{F}^1_{\varepsilon,h}(v_{\varepsilon,h}) = \int_\Omega \left[\varepsilon |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon}\omega(v_\varepsilon)\right] dx + \\ + \int_\Omega \varepsilon[|\nabla v_{\varepsilon,h}|^2 - |\nabla v_\varepsilon|^2] dx + \int_\Omega \frac{1}{\varepsilon}[\omega(v_{\varepsilon,h}) - \omega(v_\varepsilon)] dx =: I^4 + II^4 + III^4.$$

First, using that $v_\varepsilon \in D(\mathcal{F}_\varepsilon)$, we work to prove the limsup estimate for I^4. This term will be further decomposed. Let Ω be represented as

$$\Omega = (\Omega \setminus L_\varepsilon) \cup (L_\varepsilon^A \setminus T_\varepsilon) \cup \left(\bigcup_{i=1}^{m} (L_\varepsilon^{\Gamma_i} \cap A) \setminus T_\varepsilon \right) \cup \left(\bigcup_{i=1}^{m} (L_\varepsilon^{\Gamma_i} \setminus A) \setminus T_\varepsilon \right) \cup T_\varepsilon .$$

We deal with each term separately. Set $[*] := [\varepsilon |\nabla v_\varepsilon|^2 + \varepsilon^{-1} \omega(v_\varepsilon)]$. Since $v_\varepsilon = v_0$ on $\Omega \setminus L_\varepsilon$ is locally constant and $|v_0| = 1$, we have

$$I_0^4 := \int_{\Omega \setminus L_\varepsilon} [*] dx = 0. \tag{2.13}$$

In view of (2.10) and (2.6), the next term can be written in the form

$$I_1^4 := \int_{L_\varepsilon^A \setminus T_\varepsilon} [*] dx = \int_{L_\varepsilon^A \setminus T_\varepsilon} \left[\varepsilon |\gamma_\varepsilon'(d_A(x))|^2 + \frac{1}{\varepsilon} \omega(\gamma_\varepsilon(d_A(x))) \right] |\nabla d_A(x)| dx.$$

Then, using the coarea formula [12] and (1.3), we have

$$I_1^4 = \int_R \left[\varepsilon |\gamma_\varepsilon'(t)|^2 + \frac{1}{\varepsilon} \omega(\gamma_\varepsilon(t)) \right] \mathcal{H}^{n-1}(A_t \setminus T_\varepsilon) dt \leqslant$$

$$\leqslant [\mathcal{H}^{n-1}(\partial A \setminus T_\varepsilon) + o(1)] \int_{-\varepsilon\pi/2}^{\varepsilon\pi/2} \left[\varepsilon |\gamma_\varepsilon'(t)|^2 + \frac{1}{\varepsilon} \omega(\gamma_\varepsilon(t)) \right] dt = 2c_0 [\mathcal{H}^{n-1}(\partial A \setminus T_\varepsilon) + o(1)].$$

Hence

$$\limsup_{\varepsilon \to 0} I_1^4 \leqslant 2c_0 \mathcal{H}^{n-1}(\partial A \cap \Omega). \tag{2.14}$$

Similarly, for $I_{2,i}^4 := \int_{(L_\varepsilon^{\Gamma_i} \cap A) \setminus T_\varepsilon} [*] dx$ we have

$$I_{2,i}^4 = \int_0^\infty \left[\varepsilon |\gamma_\varepsilon'(t + \varepsilon \arcsin(g_i))|^2 + \frac{1}{\varepsilon} \omega(\gamma_\varepsilon(t + \varepsilon \arcsin(g_i))) \right] \mathcal{H}^{n-1}((\Omega_t \cap L_\varepsilon^{\Gamma_i} \cap A) \setminus T_\varepsilon) dt \leqslant$$

$$\leqslant [\mathcal{H}^{n-1}((\Gamma_i \cap \overline{A}) \setminus T_\varepsilon) + o(1)] \int_{\varepsilon \arcsin(g_i)}^{\varepsilon\pi/2} \left[\varepsilon |\gamma_\varepsilon'(t)|^2 + \frac{1}{\varepsilon} \omega(\gamma_\varepsilon(t)) \right] dt.$$

With the following change of variables, $\gamma_\varepsilon(t) = z$, using (2.8), (0.2), and (0.4), we obtain

$$\int_{\varepsilon \arcsin(g_i)}^{\varepsilon\pi/2} \left[\varepsilon |\gamma_\varepsilon'(t)|^2 + \frac{1}{\varepsilon} \omega(\gamma_\varepsilon(t)) \right] dt = 2 \int_{g_i}^{1} \sqrt{1 - z^2} dz = 2\delta(g_i, 1),$$

hence

$$\limsup_{\varepsilon \to 0} I_{2,i}^4 \leqslant 2\delta(1, g_i) \mathcal{H}^{n-1}(\Gamma_i \cap \partial A). \tag{2.15}$$

Arguing on $I^4_{3,i} := \int_{(L^{\Gamma_i}_\varepsilon \setminus A)\setminus T_\varepsilon} [*]dx$ as before, we obtain

$$\limsup_{\varepsilon \to 0} I^4_{3,i} \leqslant 2\delta(-1, g_i)\mathcal{H}^{n-1}(\Gamma_i \setminus \partial A). \tag{2.16}$$

Finally, (2.11) and (2.7) easily lead to

$$I^4_4 := \int_{T_\varepsilon} [*]dx = O(\varepsilon). \tag{2.17}$$

Collecting (2.13)-(2.17) and using (2.12) we thus obtain

$$\limsup_{\varepsilon \to 0} I^4 = \limsup_{\varepsilon \to 0} \left(I^4_0 + I^4_1 + \sum_{i=1}^{m} (I^4_{2,i} + I^4_{3,i}) + I^4_4 \right) \leqslant c_0 \mathcal{F}^1(v_0). \tag{2.18}$$

Our next task is to prove that terms II^4 and III^4 vanish. Since $v_{\varepsilon,h} = v_\varepsilon$ on $\Omega \setminus L_{\varepsilon,h}$, using (2.11) we get

$$|II^4| \leqslant \int_{L_{\varepsilon,h}} \varepsilon |\nabla(v_\varepsilon + v_{\varepsilon,h})| |\nabla(v_\varepsilon - v_{\varepsilon,h})| dx \leqslant C \int_{L_{\varepsilon,h}} |\nabla(v_\varepsilon - v_{\varepsilon,h})| dx.$$

Using again (2.11), in conjunction with (2.6), (2.7), (2.9), and well known properties of the interpolation operator Π_h, we obtain

$$|II^4| \leqslant \int_{T_{\varepsilon,h}} |\nabla(v_\varepsilon - v_{\varepsilon,h})| dx + \int_{L_{\varepsilon,h} \setminus T_{\varepsilon,h}} |\nabla(v_\varepsilon - v_{\varepsilon,h})| dx \leqslant \tag{2.19}$$

$$\leqslant C\varepsilon^{-1} \mathcal{H}^n(T_{\varepsilon,h}) + Ch\mathcal{H}^n(L_{\varepsilon,h} \setminus T_{\varepsilon,h}) \|D^2 v_\varepsilon\|_{L^\infty(L_\varepsilon \setminus T_\varepsilon)} \leqslant C[\varepsilon + h/\varepsilon],$$

because $D^2 v_\varepsilon = \gamma''_\varepsilon(d_A)|\nabla d_A|^2 + \gamma'_\varepsilon(d_A) D^2 d_A = \gamma''_\varepsilon(d_A)$ on $L^A_\varepsilon \setminus T_\varepsilon$ and similarly on $L^\Omega_\varepsilon \setminus T_\varepsilon$. Finally, using (2.11) and (2.7), we get

$$|III^4| = \left| \int_{L_{\varepsilon,h}} \frac{1}{\varepsilon} [\omega(v_\varepsilon) - \omega(v_{\varepsilon,h})] dx \right| \leqslant C \frac{h}{\varepsilon} \mathrm{Lip}(\omega) \mathcal{H}^n(L_{\varepsilon,h}) \|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \leqslant Ch/\varepsilon. \tag{2.20}$$

Collecting (2.18)-(2.20) and enforcing the relation $h = o(\varepsilon)$, we conclude that

$$\limsup_{\varepsilon \to 0} \mathcal{F}^1_{\varepsilon,h}(v_{\varepsilon,h}) \leqslant c_0 \mathcal{F}^1(v_0). \tag{2.21}$$

Since (2.21) guarantees that (2.1) holds, *step* 1 and *step* 3 are valid. This leads to the assertion (*ii*) and concludes the proof of the theorem. □

It is an easier task to obtain the following compactness result which, in turn, achieves the proof of Theorem 0.1, in view of basic properties of Γ-convergence.

THEOREM 2.2. *If $h = o(\varepsilon)$, then any family $\{u_{\varepsilon,h}\}_{\varepsilon,h}$ of absolute minimizers of $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon,h}$ is relatively compact in $L^1(\Omega)$.*

PROOF. We follow a standard approach (see, *e.g.*, [14, Prop. 3]). Given a family $\{u_{\varepsilon,h}\}_{\varepsilon,h}$ of absolute minimizers of $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon,h}$, it is sufficient to prove that $\{\varphi(u_{\varepsilon,h})\}_{\varepsilon,h}$ is relatively compact in $L^1(\Omega)$, where φ is defined in (0.3). We stick with the notation of

Theorem 2.1 and use (0.3) and the Young inequality to get

$$(2.22)\qquad 2\int_\Omega |\nabla\varphi(u_{\varepsilon,h})|dx = 2\int_\Omega \sqrt{\omega(u_{\varepsilon,h})}|\nabla u_{\varepsilon,h}|dx \leqslant$$

$$\leqslant \int_\Omega \left[\varepsilon|\nabla u_{\varepsilon,h}|^2 + \frac{1}{\varepsilon}\omega(u_{\varepsilon,h})\right]dx = \mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}) - c_0\mathcal{F}^2_{\varepsilon,h}(u_{\varepsilon,h}) - \mathcal{F}^3_{\varepsilon,h}(u_{\varepsilon,h}).$$

Now, let $u \in D(\mathcal{F})$ be a minimizer of $\mathcal{F}$ and $\{v_{\varepsilon,h}\}_{\varepsilon,h}$ a sequence converging to u, as of assertion (*ii*) of Theorem 2.1; then $c_0\mathcal{F}(u) \geqslant \limsup_{\varepsilon\to 0}\mathcal{F}_{\varepsilon,h}(v_{\varepsilon,h}) \geqslant \limsup_{\varepsilon\to 0}\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h})$, because $u_{\varepsilon,h}$ minimizes $\mathcal{F}_{\varepsilon,h}$. In addition, we have $|\mathcal{F}^2_{\varepsilon,h}(u_{\varepsilon,h})| \leqslant C$ (use (0.9) and $|u_{\varepsilon,h}| \leqslant 1$), and *step* 1 of Theorem 2.1 gives $\lim_{\varepsilon\to 0}\mathcal{F}^3_{\varepsilon,h}(u_{\varepsilon,h}) = 0$ $\left(\text{note that } \left\{\int_\Omega \varepsilon|\nabla u_{\varepsilon,h}|^2 dx\right\}_{\varepsilon,h} \text{ is equi-bounded with respect to } \varepsilon\right)$. Hence, inserting these estimates in (2.22), leads to the equiboundedness of $\left\{\int_\Omega |\nabla\varphi(u_{\varepsilon,h})|dx\right\}_{\varepsilon,h}$, so that $\|\varphi(u_{\varepsilon,h})\|_{BV(\Omega)} \leqslant C$, constant independent of ε. Hence the compacteness theorem in BV [18] gives the assertion. □

REMARK 2.1. Theorems 2.1 and 2.2 still hold for any graph ω such that

$$(2.23)\qquad \begin{array}{l}\omega_{|[-1,1]} \text{ is a even positive } C^{1,1} \text{ function with two isolated zeros in } -1, 1,\\ \omega = +\infty, \text{ in } \mathbf{R}\setminus[-1,1].\end{array}$$

In particular, if the following property holds:

$$\int_{-1}^{1} \frac{dt}{\sqrt{\omega(t)}} < +\infty,$$

as of (0.2), then the (new) absolute minimizer u_ε of the one dimensional functional $\mathcal{G}_\varepsilon$ in (1.1) ranges from -1 to 1 in a finite interval $O(\varepsilon)$-wide, whereas it is locally constant outside. Hence, we can still define $\gamma_\varepsilon = u_\varepsilon$ in (2.8) and even the proof of Theorems 2.1 and 2.2 remains unchanged. On the contrary, if ω verifies (2.23), but

$$\int_{-1}^{1} \frac{dt}{\sqrt{\omega(t)}} = +\infty,$$

then there is unicontinuation, that is the solution of (1.2) cannot assume the values -1 and 1 in $\mathbf{R}$. As an example, consider the classical function $\omega(t) := (1-t^2)^2$; then the solutions of (1.2) are $v_\varepsilon(x) = \tanh[(x-a)/\varepsilon]$. In order to prove the constructive part (*ii*) of Theorem 2.1, we can define the $W^{2,\infty}$ even function $\gamma_\varepsilon(t)$ in (2.8) as follows: $\gamma_\varepsilon(t) := v_\varepsilon(t)$, if $t \in [0, \varepsilon|\log\varepsilon|]$ (set $a = 0$); $\gamma_\varepsilon(t) = 1$, if $t \geqslant 2\varepsilon|\log\varepsilon|$; $\gamma_\varepsilon \in \mathbf{P}^3(\varepsilon|\log\varepsilon|, 2\varepsilon|\log\varepsilon|)$; $\gamma_\varepsilon(t) = -\gamma_\varepsilon(-t)$, if $t < 0$. It is easy to check that the property (2.9) still holds and that $\mathcal{G}_\varepsilon(\gamma_\varepsilon) \leqslant \mathcal{G}_\varepsilon(u_\varepsilon) + C\varepsilon^4$, where u_ε is an absolute minimizer of $\mathcal{G}_\varepsilon$. Since the *transition* interval of γ_ε has now size $4\varepsilon|\log\varepsilon|$, the definition of the sets L^A_ε, L^Ω_ε, and T_ε has to be modified, as well. Assertion (*ii*) of Theorem 2.1 then follows with the fairly stronger assumption $h = o(\varepsilon|\log\varepsilon|^{-1})$.

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