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Inverse Fourier transform

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Analisi funzionale. — *Inverse Fourier Transform.* Nota di LEONEDE DE MICHELE, MARINA DI NATALE e DELFINA ROUX, presentata (*) dal Socio L. AMERIO.

ABSTRACT. — In this paper a very general method is given in order to reconstruct a periodic function f knowing only an approximation of its Fourier coefficients.

KEY WORDS: Fourier series; Noisy data; Inversion.

RIASSUNTO. — *Sull'inversione della trasformata di Fourier.* In questa *Nota* si presenta un metodo molto generale per risolvere il problema mal posto di ricostruire una funzione periodica quando siano noti soltanto valori approssimati dei suoi coefficienti di Fourier.

1. In the applications, a serious trouble in using Fourier techniques is that the Fourier transform has not in general a continuous inverse. This implies, that except for L^2, the reconstruction of a periodic function f with approximated Fourier coefficients is not a well-posed problem.

Tikhonov and Arsénine (see [6]), using the notion of regularizing operator and variational techniques, suggested a quite general method to solve the above problem for particular subclasses of continuous functions.

For more general families of functions, using Fourier Analysis tools, we constructed in [1] some classes of regularizing operators which turn out to be of interest also for numerical applications (see *e.g.* [2-5]).

In this paper we show that the method of [1] can be developed in a very general setting, using approximate identity techniques.

2. Let us now introduce some standard notations.

If $N \geq 1$, let $\boldsymbol{Z}^N$ be the lattice of integer points of $\boldsymbol{R}^N$ and $\boldsymbol{T}^N = \boldsymbol{R}^N/\boldsymbol{Z}^N$ the N-dimensional torus. Let us set B, indifferently, the Lebesgue space $L^p(\boldsymbol{T}^N)$, $1 \leq p < +\infty$, or the space of continuous functions $C(\boldsymbol{T}^N)$ and denote their norm by $\| \ \|_B$. For convenience, we identify $\boldsymbol{T}^N$ with $[-1/2, 1/2)^N = Q^N$.

Let $p_0 = \min(p, 2)$ (for $B = C(\boldsymbol{T}^N)$ we set $p = +\infty$) and q_0 such that $1/p_0 + 1/q_0 = 1$ and let $b = l^{q_0}(\boldsymbol{Z}^N)$ with the usual norm.

If $f \in L^1(\boldsymbol{T}^N)$ or $L^1(\boldsymbol{R}^N)$ we denote by $\hat{f}$ respectively the sequence of its Fourier coefficients

$$\hat{f}(n) = \int_{\boldsymbol{T}^N} f(t)\, e^{-2\pi i n t}\, dt\,, \qquad n \in \boldsymbol{Z}^N$$

or its Fourier transform

$$\hat{f}(x) = \int_{\boldsymbol{R}^N} f(y)\, e^{-2\pi i x y}\, dy\,, \qquad x \in \boldsymbol{R}^N .$$

(*) Nella seduta del 14 giugno 1990.

3. If $G \in L^1(\boldsymbol{R}^N)$ such that $\hat{G}(0) = 1$, we set $G_\sigma(x) = \sigma^{-N} G(x/\sigma)$, $\sigma > 0$ and, for every $\lambda = \{\lambda_n\}_{n \in \boldsymbol{Z}^N}$, $\lambda_n \in \boldsymbol{C}$ $\forall n$

$$R_\sigma \lambda \sim \sum_{n \in \boldsymbol{Z}^N} \hat{G}(\sigma n) \lambda_n e^{2\pi i n t}, \qquad t \in \boldsymbol{T}^N. \tag{3.1}$$

The following theorem holds.

THEOREM 1. *If* $\{\hat{G}(\sigma n)\} \in l^{p_0}(\boldsymbol{Z}^N)$ *for every* $\sigma > 0$, *and* $\sigma = \sigma(\delta)$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ *satisfies*

$$\lim_{\delta \to 0} \sigma(\delta) = 0, \qquad \lim_{\delta \to 0} \delta \|\{\hat{G}(\sigma(\delta)n)\}\|_{l^2}^{|1-2/p|} = 0, \tag{3.2}$$

then, if $f \in B$, *for every* $\varepsilon > 0$ *there exists* $\delta_0 = \delta_0(\varepsilon, f)$ *such that if* $\delta \leqq \delta_0$

$$\lambda \in b \wedge \|\lambda - \hat{f}\|_b \leqq \delta \Rightarrow \|f - R_{\sigma(\delta)} \lambda\|_B < \varepsilon. \tag{3.3}$$

For instance: $p > 2$, $\delta \leqslant \delta_0$, $\lambda \in l^2 \wedge \|\lambda - \hat{f}\|_{l^2} \leqslant \delta \Rightarrow \|f - R_{\sigma(\delta)} \lambda\|_{L^p} < \varepsilon$.

Since $\|\{\hat{G}(\sigma n)\}\|_{l^2} \to \infty$ if $\sigma \to 0^+$, the theorem shows that the approximation of f by $R_\sigma \lambda$ is substantially controlled by the behaviour of $\|\{\hat{G}(\sigma n)\}\|_{l^2}$.

In many cases of interest it is possible to give sharp evaluations of the above norm.

An *a priori* estimate of the error $\|f - R_\sigma \lambda\|_B$ is given in Liptschitz classes of functions by the Theorem 2.

We denote, as usual, by $K \operatorname{Lip}(\alpha, B)$, $0 < \alpha \leqslant 1$, the class of $f \in B$ such that for every $u \in \boldsymbol{T}^N$ the function $\Delta_u f(t) = f(t+u) - f(t)$ satisfies the condition $\|\Delta_u f\|_B \leqslant K|u|^\alpha$.

Then if $\mu_{f,B}(\delta, \sigma) = \operatorname{Sup}\{\|f - R_\sigma \lambda\|_B : \lambda \in b \wedge \|\hat{f} - \lambda\|_b \leqq \delta\}$ we have

THEOREM 2. *If* $\{\hat{G}(\sigma n)\} \in l^{p_0}(\boldsymbol{Z}^N)$ *for every* $\sigma > 0$ *and*

$$\int_{\boldsymbol{R}^N} |x|^\alpha |G(x)|\, dx = c_\alpha < +\infty,$$

then, for every $\delta > 0$, $\sigma > 0$, *if* $f \in K \operatorname{Lip}(\alpha, B)$, *we have*

$$\mu_{f,B}(\delta, \sigma) \leqslant K c_\alpha \sigma^\alpha + a_p \|\{\hat{G}(\sigma n)\}\|_{l^2}^{|1-2/p|} \delta, \tag{3.4}$$

where $a_p = \|G\|_{L^1}^{1-|1-2/p|}$.

We remark that if $G(x) \geqslant 0$ *a.e.* on $\boldsymbol{R}^N$, $\delta = 0$ and $B = C(\boldsymbol{T}^N)$ then the inequality (3.4) is sharp.

4. In this section we point out the aspect of pointwise convergence of the above procedure. Obviously, in order to do this, we have to restrict ourselves to the Lebesgue points of f and, as usual, we have to make some additional request on the function G.

Let us set $M(x) = \operatorname{Sup\,ess}_{|y| \geqslant |x|} |G(y)|$.

THEOREM 3. *If* $\{\hat{G}(\sigma n)\} \in l^{p_0}(\boldsymbol{Z}^N)$ *for every* $\sigma > 0$, $M \in L^1(\boldsymbol{R}^N)$ *and*

$\sigma = \sigma(\delta) : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ *satisfies*

$$\text{(4.1)} \qquad \lim_{\delta \to 0} \sigma(\delta) = 0\,, \qquad \lim_{\delta \to 0} \delta \| \{ \hat{G}(\sigma(\delta)n) \} \|_{p_0} = 0\,,$$

then, if f $\in B$ *and* t *is a Lebesgue point of* f, *for every* $\varepsilon > 0$ *there exists* $\delta_0 = \delta_0(\varepsilon, \mathrm{f}, \mathrm{t})$ *such that if* $\delta \leqq \delta_0$

$$\text{(4.2)} \qquad \lambda \in b \wedge \| \lambda - \hat{f} \|_b \leqq \delta \Rightarrow |f(t) - R_{\sigma(\delta)} \lambda(t)| < \varepsilon\,.$$

An *a priori* estimate of the error in (4.2) can be given in suitable subspaces of B, strictly related to the Morrey-Campanato classes (see [1]). We briefly recall the definition.

A function $f \in L^1(\boldsymbol{T}^N)$ belongs to a class K Leb (α, t), $0 < \alpha \leqslant 1$, $t \in \boldsymbol{T}^N$ if for every r, $0 < r \leqslant \sqrt{N}/2$ we have

$$\int_{|x-t| \leqslant r} |f(x) - f(t)|\, dx \leqslant K r^{N+\alpha}\,.$$

Now, if $f \in B$ and $\mu_{f,t}(\delta, \sigma) = \operatorname{Sup} \{ |f(t) - R_\sigma \lambda(t)| : \lambda \in b \wedge |\lambda - \hat{f}| \leqslant \delta \}$ we have

Theorem 4. *If* $\{\hat{G}(\sigma n)\} \in l^{p_0}(\boldsymbol{Z}^N)$ *for every* $\sigma > 0$ *and*

$$\int_{\boldsymbol{R}^N} |x|^\alpha M(x)\, dx = \gamma_\alpha < +\infty,$$

then for every $\delta > 0$, $0 < \sigma \leqslant \sigma_0$, *if* $f \in B \cap K \operatorname{Leb}(\alpha, t)$ *we have*

$$\text{(4.3)} \qquad \mu_{f,t}(\delta, \sigma) \leqslant K \bar{c}_\alpha \sigma^\alpha + \delta \| \{ \hat{G}(\sigma n) \} \|_{p_0}$$

where $\bar{c}_\alpha = \gamma_\alpha (N + \alpha + 1)/m(S_n)$ *and* $m(S_n)$ *is the surface area* of *the unit ball of* $\boldsymbol{R}^N$.

We remark that if $G(x) \geqslant 0$ *a.e.* on $\boldsymbol{R}^N$ and $B = C(\boldsymbol{T}^N)$, it is not possible to get, for the all class of functions, a better behaviour with respect to σ.

5. Of course, in order to apply this method, not only we have to consider functions G which satisfy the hypotheses but also we have to estimate the behaviour with respect to σ of $\| \{ \hat{G}(\sigma n) \} \|$ in $l^p(\boldsymbol{Z}^N)$ spaces $(1 \leq p \leq 2)$.

For instance, it is easy to do this for classical kernels. For other classes (*e.g.* that ones of [1]) it is more convenient to start the method from the Fourier transform side.

In a forthcoming paper, together with the proofs of the theorems, we show some large and interesting classes of functions G where all the estimates we need, for making the method effective, can be easily carried out.

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