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Asymptotic behaviour in planar vortex theory

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Analisi matematica. — *Asymptotic behaviour in planar vortex theory.* Nota di ANTONIO AMBROSETTI e YANG JIANFU, presentata (*) dal Corrisp. A. AMBROSETTI.

ABSTRACT. — The asymptotic behaviour of solutions of a class of free-boundary problems arising in vortex theory is discussed.

KEY WORDS: Free boundary problems; Vortex theory; Nonlinear desingularization.

RIASSUNTO. — *Comportamento asintotico nella teoria dei vortici.* Viene discusso il comportamento asintotico delle soluzioni di certi problemi di frontiera libera che intervengono nella teoria dei vortici.

1. INTRODUCTION

Consider an inviscid fluid with uniform density, confined in a bounded subset Ω of $\boldsymbol{R}^2$. The existence of a «vortex» in such a fluid can be formulated as a free boundary problem, seeking an open «vortex core» $A \subset \Omega$ and a stream function $\Psi \in C^1(\Omega) \cap C^2(\Omega \setminus \partial A)$ satisfying

$$(1\lambda) \qquad \begin{cases} -\Delta \Psi = \lambda f(\Psi) \ \text{ in } A \\ -\Delta \Psi = 0 \ \text{ in } \Omega / \overline{A} \\ \Psi_{|\partial A} = 0 \ \text{ and } \ \Psi > 0 \ \text{ in } A \\ \Psi = -\Psi_0 < 0 \ \text{ on } \partial\Omega \end{cases}$$

where Ψ_0 and the vorticity function f are given. The corresponding solution pair of (1λ) will be denoted by $(\Psi_\lambda, A_\lambda)$.

Under the assumption that f is «superlinear» at infinity we will study the limiting behaviour as $\lambda \to \infty$ of the vortex core A_λ and the stream function Ψ_λ. We will show that the diameter of the vortex core tends to 0 as $\lambda \to \infty$; moreover, Ψ_λ converges to a function with an isolated singularity.

Our results are related to those of [4] which, actually, deal with a different problem because the parameter λ is not prescribed but arises as a Lagrange multiplier.

In section 2 we recall an existence result for (1λ). The limiting behaviour of the solution pair $(\Psi_\lambda, A_\lambda)$ as $\lambda \to \infty$ is studied in section 4. Our proof relies on some estimates of the H^1 norm of Ψ_λ and of the diam (A_λ), given in section 3.

2. EXISTENCE RESULTS

Existence results in vortex theory are well known: see, for example [1-3, 6-8] dealing with vortex rings in a cylindrically simmetric fluid filling all of $\boldsymbol{R}^3$, and [9]

(*) Nella seduta del 14 giugno 1990.

for planar vortex pairs. Similar arguments apply in the case of (1λ). In particular we will refer to the method developed in [1, §2] to get the following result.

THEOREM 1. *Let* $\Psi_0 > 0$ *on* $\partial\Omega$ *be smooth and suppose* f *satisfies:*

($f1$) $f \in C^2(\mathbf{R}^+, \mathbf{R})$, $f(0) = 0$, $f(s) > 0 \ \forall s > 0$, *and* $f(s) \leqslant c_1 + c_2 s^p$, *for some* $c_1, c_2, p > 0$;

($f2$) $\exists \theta \in (0, 1/2)$ *such that* $F(s) \leqslant \theta s f(s) \ \forall s \geqslant 0$ *where* $F(s) := \int_0^s f(\sigma)\, d\sigma$;

($f3$) f *is strictly convex and increasing.*

Then for all $\lambda > 0$, (1λ) *has a solution* $(\Psi_\lambda, A_\lambda)$. *Furthermore,* $A_\lambda = \{\Psi_\lambda(x) \in \Omega : \Psi_\lambda(x) > 0\}$ *is connected.*

Although the proof of Theorem 1 is similar to that in [1, §2], it is convenient to give an outline for future references. Let $q(x)$ be the solution of

$$\begin{cases} -\Delta q = 0 & \text{on } \Omega \\ q = \Psi_0 & \text{on } \partial\Omega . \end{cases}$$

By the maximum principle $K_0 := \min\{q(x) : x \in \overline{\Omega}\} > 0$. Let us extend $f(s)$ to all $\mathbf{R}$ by setting $f(s) \equiv 0$ for $s < 0$ (in the sequel we will use the same symbol f to denote such an extension), and let us look for positive solutions $\psi = \psi(x)$ of

$$(P\lambda) \qquad \begin{cases} -\Delta\psi = \lambda f(\psi - q) & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega . \end{cases}$$

If ψ is such a solution then $\Psi = \psi - q$ solves (1λ).

For $\psi \in H_0^1(\Omega)$ let $\|\psi\|^2 = \int_\Omega |\nabla\psi|^2\, dx$ and $I_\lambda(\psi) = 1/2\|\psi\|^2 - \lambda \int_\Omega F(\psi - q)\, dx$.

Critical points of I_λ correspond to positive solutions ψ_λ of (P_λ). In order to find critical points of I_λ suitable for the limiting procedure, one seeks the minimum of I_λ constrained on

$$M(\lambda) = \{\psi \in H_0^1(\Omega) \setminus \{0\} : g(\psi) = \|\psi\|^2 - \lambda \int_\Omega \psi f(\psi - q)\, dx = 0\} .$$

Under our assumptions one shows that: (*i*) for all $\phi \in H_0^1(\Omega)$, $\phi > 0$ the function $\gamma(t) := t^{-1} g(t\phi)$ is strictly decreasing and the ray $\{t\phi\}_{t>0}$ meets transversally $M(\lambda)$ in exactly one point; (*ii*) hence $M(\lambda)$ is a smooth submanifold of $H_0^1(\Omega)$; (*iii*) if $\psi \in M(\lambda)$, $t \to I_\lambda(t\psi)$ is increasing for $t \in [0, 1]$; (*iv*) I_λ achieves the minimum at some $\psi_\lambda \in M(\lambda)$; and (*v*) $\operatorname{grad} I_\lambda(\psi_\lambda) = 0$. Moreover, using the fact that ψ_λ is the minimum of I_λ on $M(\lambda)$, one shows that the vortex core $A_\lambda = \{\psi_\lambda > q\}$ is connected, see theorem 4 of [1].

3. PRELIMINARY LEMMAS

In the sequel we shall need to compare (P_λ) with similar problems involving suitable subsets D of Ω, as well as the boundary value q_0 and a «model» nonlinearity like

t^m. To point out such a dependence, we will set

$$M(\lambda, D, f, q) = \left\{ \psi \in H_0^1(D) : \int_D |\nabla \psi|^2 \, dx = \lambda \int_D \psi f(\psi - q) \, dx \right\}.$$

Similarly, we indicate by $I_{\lambda, D, f, q}$ the functional corresponding to I_λ, $P(\lambda, D, f, q)$ the variational problem $\min\{I_{\lambda,D,f,q}(\psi) : \psi \in M(\lambda, D, f, q)\}$ and $C(\lambda, D, f, q) = \min\{I_{\lambda,D,f,q}(u) : u \in M(\lambda, D, f, q)\}$. By (*f*1-3) there exists a constant $c_0 > 0$ such that, letting $m = (1-\theta)/\theta \geq 1$ and $f_1(t) = c_0 t^m$, one has $f(t) \geq f_1(t)$, for all $t \geq 0$.

We start showing:

LEMMA 2. *Let B be a fixed ball contained in Ω and let $q_0 = max\{q(x) : x \in \overline{\Omega}\}$. Then one has: $C(\lambda, \Omega, f, q) \leq C(\lambda, B, f_1, q_0)$.*

PROOF. First we claim that:

$$C(\lambda, \Omega, f, q) \leq C(\lambda, \Omega, f, q_0) \tag{2}$$

To prove (2), let ψ_0 be a solution of $P(\lambda, \Omega, f, q_0)$. Since f is strictly increasing, then

$$0 = \|\psi_0\|^2 - \lambda \int_\Omega \psi_0 f(\psi_0 - q_0) \, dx \geq \|\psi_0\|^2 - \lambda \int_\Omega \psi_0 f(\psi_0 - q) \, dx.$$

Since $\gamma(t)$ is strictly decreasing, there exists $t_0 \in (0,1)$ such that $t_0 \psi_0 \in M(\lambda, \Omega, f, q)$ and this yields $C(\lambda, \Omega, f, q) \leq I_{\lambda,\Omega,f,q}(t_0 \psi_0)$. Since $I_{\lambda,\Omega,f,q}$ is increasing with respect to q, then $C(\lambda, \Omega, f, q) \leq I_{\lambda,\Omega,f,q_0}(t_0 \psi_0)$. In addition, since $t \to I_{\lambda,\Omega,f,q}(t\phi)$ is increasing for $t \in [0,1]$, then $I_{\lambda,\Omega,f,q_0}(t_0 \psi_0) < I_{\lambda,\Omega,f,q_0}(\psi_0)$ and (2) follows.

Next, we show:

$$C(\lambda, \Omega, f, q_0) \leq C(\lambda, B, f, q_0). \tag{3}$$

To see this, first let φ be a solution of the problem $P(\lambda, B, f, q_0)$. Extend φ to ψ_B in $H_0^1(\Omega)$ by setting $\psi_B = 0$ outside B; then $\psi_B \in M(\lambda, \Omega, f, q_0)$ and $C(\lambda, \Omega, f, q_0) \leq I_{\lambda,\Omega,f,q_0}(\psi_B) \leq I_{\lambda,B,f,q_0}(\varphi) = C(\lambda, B, f, q_0)$.

Lastly, let ψ_1 be a solution of $P(\lambda, B, f_1, q_0)$. Since $f \geq f_1$, we have

$$\int_B |\nabla \psi_1|^2 \, dx - \lambda \int_B \psi_1 f(\psi_1 - q_0) \, dx \leq 0.$$

So, there exists $t_1 \in (0,1)$ such that $t_1 \psi_1 \in M(\lambda, B, f, q_0)$ and as before one has $C(\lambda, B, f, q_0) \leq C(\lambda, B, f_1, q_0)$. This, jointly with (2) and (3) proves the lemma. Q.E.D.

To estimate $C(\lambda, B, f_1, q_0)$ we consider a ball $B \subset \Omega$ centered in x_0 with radius b and set $r = |x - x_0|$.

LEMMA 3. *If B is as before, then $C(\lambda, B, f_1, q_0) \to 0$ as $\lambda \to \infty$.*

PROOF. Setting $K = 5(m+1)/c_0$, it is easy to check (recall that $m \geq 1$) that, for λ large enough there exists, in a deleted neighbourhood of $a = 0$, an unique $a = a_\lambda$

satisfying

$$a^2\,[q_0\,(2\log(b/a))^{-1}]^{m-1} = K\lambda^{-1}. \tag{4}$$

We put $\sigma_\lambda = 1/\log(b/a_\lambda)$, $\alpha_\lambda = q_0\,\sigma_\lambda/2$ and

$$\phi_\lambda(r) = \begin{cases} \alpha_\lambda\,(1-(r/a_\lambda)^2) & \text{for } 0 \leq r \leq a_\lambda \\ -q_0\,\sigma_\lambda \log(r/a_\lambda) & \text{for } a_\lambda \leq r \leq b\,. \end{cases}$$

Let us note explicitely that ϕ' is continuous at $r = a_\lambda$. Moreover, we remark that a_λ, σ_λ and $\alpha_\lambda \to 0$ as $\lambda \to \infty$.

Set $u_\lambda(x) = \phi_\lambda(|x|) + q_0$. With direct calculations one finds:

$$\int_B |\nabla u_\lambda|^2\,dx = 2\pi(\alpha_\lambda^2 + q_0^2\,\sigma_\lambda) = 2\pi(\alpha_\lambda^2 + 2q_0\,\alpha_\lambda)\,;$$

$$\lambda c_0 \int_{\{u_\lambda \geq q_0\}} (u_\lambda - q_0)^m\,u_\lambda\,dx = 2\pi\lambda c_0 \int_0^{a_\lambda} \phi\lambda^m\,(\phi_\lambda + q_0)\,r dr =$$

$$= \pi\lambda c_0\,a_\lambda^2\,\alpha_\lambda^m\,(\alpha_\lambda\,(m+2)^{-1} + q_0\,(m+1)^{-1}) = \pi c_0\,K\alpha_\lambda\,(\alpha_\lambda\,(m+2)^{-1} + q_0\,(m+1)^{-1})\,.$$

As a consequence, as $\lambda \to \infty$ one has that

$$\frac{1}{\alpha_\lambda}\int_B |\nabla u_\lambda|^2\,dx \to 4\pi q_0 \tag{5}$$

$$\frac{\lambda c_0}{\alpha_\lambda}\int_{\{u_\lambda \geq q_0\}} (u_\lambda - q_0)^m\,u_\lambda\,dx \to \pi c_0\,K q_0\,(m+1)^{-1} = 5\pi q_0\,. \tag{6}$$

From (5) and (6) it follows that for λ large enough there results:

$$\int_B |\nabla u_\lambda|^2\,dx < \lambda c_0 \int_{\{u_\lambda \geq q_0\}} (u_\lambda - q_0)^m\,u_\lambda\,dx.$$

Then there exists $t_\lambda < 1$ such that $t_\lambda\,u_\lambda \in M_{\lambda,B,f_1,q_0}$ and hence

$$C(\lambda,B,f_1,q_0) \leq I_{\lambda,B,f_1,q_0}(t_\lambda\,u_\lambda) < I_{\lambda,B,f_1,q_0}(u_\lambda) \leq \frac{1}{2}\int_B |\nabla u_\lambda|^2\,dx = \pi(\alpha_\lambda^2 + 2q_0\,\alpha_\lambda)\,. \tag{7}$$

Since, as remarked before, $\alpha_\lambda \to 0$ as $\lambda \to \infty$, then $C(\lambda,B,f_1,q_0) \to \infty$ as $\lambda \to \infty$, as required. Q.E.D.

We can now prove the main result of this section:

LEMMA 4. *Let* $C(\lambda) = \mathrm{Min}\,\{I_\lambda(u) : u \in M_\lambda\}$ *and let* ψ_λ *be a solution of* (P_λ). *Then:*

(*i*) $C(\lambda) \to 0$ *as* $\lambda \to \infty$; (*ii*) $\|\psi_\lambda\| \to 0$ *as* $\lambda \to \infty$.

PROOF. (*i*) follows directly from lemmas 2 and 3.

(*ii*) From (*f*2) it follows that

$$C(\lambda) = 1/2\|\psi_\lambda\|^2 - \lambda \int_\Omega F(\psi_\lambda - q)\, dx \geq 1/2\|\psi_\lambda\|^2 - \theta\lambda \int_\Omega f(\psi_\lambda - q)\, \psi_\lambda\, dx . \tag{8}$$

Since $\psi_\lambda \in M_\lambda$ then one finds $C(\lambda) \geq (1/2 - \theta)\|\psi_\lambda\|^2$ and the result follows from (*i*). Q.E.D.

4. Limiting behaviour of A_λ and Ψ_λ

We are now in position to study the asymptotic behaviour of the solution pair $(A_\lambda, \Psi_\lambda)$. Our main results are:

Theorem 5. *Let $\Psi_0 > 0$ on $\partial\Omega$ be smooth and suppose f satisfies* (f1-2-3). *Then:*

(*i*) diam $A_\lambda \to 0$ *as* $\lambda \to \infty$.

Theorem 6. *Let $\Psi_0 > 0$ on $\partial\Omega$ be smooth and suppose f satisfies* (f1-2-3). *Let Ψ_λ be the solution of* (P_λ) *obtained in Theorem* 1, *and define*

$$h(\lambda) = \lambda \int_{A_\lambda} f(\psi_\lambda - q)\, dx .$$

Then, for any point $\xi(\lambda) \in A_\lambda$, we have $\psi_\lambda(\cdot)/h(\lambda) - G(\cdot, \xi(\lambda)) \to 0$ in $H_0^{1,p}(\Omega)$ $1 \leq p < 2$, as $\lambda \to \infty$, where G is the Green function of $-\Delta$ in Ω.

The proofs of the preceding theorems rely on some arguments of [4, 5] which can be carried out in the present situation because of Lemma 4 before. To make the paper as selfcontained as possible we will outline the proofs.

Proof of theorem 5. The argument is similar to that of Lemma 3.1 of [5]. Let $P, Q \in \overline{A}_\lambda$ be such that $|P - Q| = \text{diam}(A_\lambda)$ and consider a family of straight lines l_X passing through the point $X \in [P, Q]$ and orthogonal to $[P, Q]$. Denote by $L_X = [Y_X, Z_X]$ a segment in l_X such that $Y_X \in \partial\Omega$, $Z_X \in \partial A_\lambda$ and $\text{int}(L_X) \subset \Omega \setminus \overline{A}_\lambda$. Then one has

$$\psi_\lambda(Y_X) - \psi_\lambda(Z_X) = \int_{L_X} \frac{\partial \psi_\lambda}{\partial L_X}\, dL_X .$$

Note that $\psi_\lambda(Y_X) = 0$ while $\psi_\lambda(Z_X) = q(Z_X) \geq K_0 > 0$. Then we infer:

$$K_0 \leq \left| \int_{L_X} \frac{\partial \psi_\lambda}{\partial L_X}\, dL_X \right| \leq c_1 \int_{L_X} |\nabla \psi_\lambda|\, dL_X .$$

Integrating with respect to X in $[P, Q]$ and using the Hölder inequality, we find readily:

$$K_0 |P-Q| \leq c_1 \int_{PQ} dX \int_{L_X} |\nabla\psi_\lambda| \, dL_X \leq c_2 |P-Q|^{1/2} \|\psi_\lambda\| .$$

The proof now follows from Lemma 4-(*ii*).

Proof of theorem 6. We follow the arguments of Theorem 5.2 of [4]. We know that

$$\psi_\lambda(z) = \lambda \int_{A_\lambda} G(z,x) f(\psi_\lambda - q) \, dx ; \qquad \frac{\lambda}{b(\lambda)} \int_{A_\lambda} f(\psi_\lambda - q) \, dx = 1 .$$

Then for $\xi(\lambda) \in A_\lambda$ one has:

$$\psi_\lambda(z)/b(\lambda) - G(z, \xi(\lambda)) = \frac{\lambda}{b(\lambda)} \int_{A_\lambda} \{G(z,x) - G(z,\xi(\lambda))\} f(\psi_\lambda - q) \, dx .$$

By the Minkowski inequality there results

$$(9) \quad \|\psi_\lambda(\cdot)/b(\lambda) - G(\cdot, \xi(\lambda))\|_{1,p,\Omega} \leq \frac{\lambda}{b(\lambda)} \int_{A_\lambda} f(\psi_\lambda - q) \, dx \left[\int_\Omega |\nabla_Z \{G(z,x) - G(z,\xi(\lambda))|^p \, dz \right]^{1/p} .$$

Lemma 5.1 of [4] yields:

$$(10) \quad \int_\Omega |\nabla_Z \{G(z,x) - G(z,\xi(\lambda))|^p \, dz \leq c_1 |x - \xi(\lambda)|^p (1 + \log(\operatorname{diam}\Omega / |x - \xi(\lambda)|))^2 .$$

Since x and $\xi(\lambda)$ are both in A_λ then $|x - \xi(\lambda)| \leq \operatorname{diam}(A_\lambda)$ and the conclusion follows from (9), (10) and Theorem 5. Q.E.D.

Remarks. (*i*) For applications, it can be useful, to state explicitely an asymptotic estimate of $\|\psi_\lambda\|$. According to (7) and (8), $\|\psi_\lambda\| \leq c_1(\alpha_\lambda^2 + \alpha_\lambda)$, where $\alpha_\lambda \cong (\log(1/s))^{-1}$, and $s = a_\lambda / b$ solves (see [4]) $s[\log(1/s)]^{-(m-1)/2} = k\lambda^{-1/2}$ for a suitable positive constant k. It is easy to check (see Lemma C2 of [4]) that $1/s \geq \vartheta(\lambda) := \sqrt{\lambda}(\log\sqrt{\lambda})^{-(m-1)/2}$ and hence $\alpha_\lambda \cong (\log(1/s))^{-1} \leq 1/\log\vartheta(\lambda)$. This provides an upper bound for $\|\psi_\lambda\|$ in terms of λ as $\lambda \to \infty$. In a similar way one can find a lower bound for $\|\psi_\lambda\|$.

(*ii*) The same arguments apply to any free boundary problem like

$$\begin{cases} -Lu = \lambda f(u-q) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $q > 0$ in Ω and L is an uniformly elliptic variational second order operator with smooth coefficients.

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