Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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The equations of viscous incompressible nonhomogeneous fluids in noncylindrical domains: on the existence and regularity

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **1** (1990), n.4, p. 281–284.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1990_9_1_4_281_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1990.

Rend. Mat. Acc. Lincei
s. 9, v. 1:281-284 (1990)

Analisi matematica. — *The equations of viscous incompressible non-homogeneous fluids in non-cylindrical domains: On the Existence and Regularity.* Nota di RODOLFO SALVI, presentata (*) dal Socio L. AMERIO.

ABSTRACT. — We prove the existence of a weak solution and of a strong solution (locally in time) of the equations which govern the motion of viscous incompressible non-homogeneous fluids. Then we discuss the decay problem.

KEY WORDS: Non-homogeneous fluids; Time dependent domains; Weak solutions; Strong solutions.

RIASSUNTO. — *Le equazioni dei fluidi viscosi incomprimibili non omogenei in domini non cilindrici: esistenza e regolarità.* Si dimostra l'esistenza di una soluzione debole e di una soluzione forte (in piccolo) per le equazioni che governano il moto dei fluidi viscosi incomprimibili con densità non costante. Inoltre si discute il problema dell'andamento asintotico.

1. INTRODUCTION

We consider the motion of a viscous incompressible non-homogeneous fluid, defined in a domain with moving boundaries. In other words, we have to deal not with a space-time cylinder but with a non-cylindrical domain in $\mathbf{R}^3 \times [0, T]$. To be more precise, we consider a domain $\Omega_T = \bigcup_{0 \leq t \leq T} \Omega(t) \times \{t\}$ where each $\Omega(t)$ is a bounded domain in $\mathbf{R}^3$, and $T > 0$ is a positive number. We will find, in the region Ω_T a solution (u, ρ, p) of the system

$$\rho \partial_t u - \mu \Delta u + \rho u \cdot \nabla u + \nabla p - \rho f = 0 ; \quad \partial_t \rho + u \cdot \nabla \rho = 0; \quad \nabla \cdot u = 0 \quad \text{in } \Omega_T \tag{1.1}$$

satisfying the initial-boundary conditions

$$\begin{cases} u(x, 0) = u_0 ; & \rho(0) = \rho_0 \quad \text{in } \Omega(0), \\ u(x, t) = 0 & \text{on } \Gamma_T, \end{cases} \tag{1.2}$$

where $u = u(t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ is the velocity, $\rho = \rho(t) = \rho(x, t)$ the density, $p = p(t) = p(x, t)$ the pressure, $f = f(t) = (f_1(x, t), f_2(x, t), f_3(x, t))$ the external force, μ the viscosity, and $\Gamma_T = \bigcup_{0 \leq t \leq T} \Gamma(t) \times \{t\}$ with $\Gamma(t)$ the boundary of $\Omega(t)$.

Problem (1.1), (1.2) was studied in [1, 2, 5], in cylindrical domains. The paper deals with the existence of weak solutions and of strong (locally in time) solutions of (1.1), (1.2). To prove this, we employ the method developed in [4].

Section 2 contains preliminaries. Section 3 contains the proof of the existence of a weak solution and of a strong (locally in time) solution of (1.1), (1.2), and contains results on the decay problem.

(*) Nella seduta del 21 aprile 1990.

2. Preliminaries

All functions in this paper are $\boldsymbol{R}$- or $\boldsymbol{R}^3$-valued. The letter c denotes different constants depending on Ω_T and α, β are positive constants. We employ the usual notations of vector analysis; in particular, the j-th components of $u \cdot \nabla u$ and Δu are

$$\sum_{i=1}^{3} u_i\, \partial_{x_i} u_j \quad \text{and} \quad \sum_{i=1}^{3} \partial_{x_i} \partial_{x_i} u_j,$$

respectively. Some additional notation is needed. We let

$$(u, v)_{\Omega(t)} = \sum_{i=1}^{3} \int_{\Omega(t)} u_i\, v_i\, dx\,; \qquad |u|^2_{\Omega(t)} = (u, u)_{\Omega(t)}\,; \qquad ((u, v))_{\Omega(t)} = \sum_{i=1}^{3} \int_{\Omega(t)} \nabla u_i\, \nabla v_i\, dx\,;$$

$$\|u\|^2_{\Omega(t)} = ((u, u))_{\Omega(t)}\,; \qquad |u|^2_{\Omega_T} = \int_0^T |u|^2_{\Omega(t)}\, dt\,; \qquad \|u\|^2_{\Omega_T} = \int_0^T \|u\|^2_{\Omega(t)}\, dt\,;$$

$D(\Omega(t)) = \{\phi \mid \phi \in (C_0^\infty(\Omega(t)))^3,\ \nabla \cdot \phi = 0\}$; $D(\Omega_T) = \{\phi \mid \phi \in (C^\infty(\Omega_T))^3, \operatorname{supp}\phi \subset \Omega_T, \nabla \cdot \phi = 0\}$; $H(\Omega(t))$ = completion of $D(\Omega(t))$ in the norm $|\phi|_{\Omega(t)}$; $V(\Omega(t))$ = completion of $D(\Omega(t))$ in the norm $\|\phi\|_{\Omega(t)}$; $H(\Omega_T)$ = completion of $D(\Omega_T)$ in the norm $|u|_{\Omega_T}$; $V(\Omega_T)$ = completion of $D(\Omega_T)$ in the norm $\|u\|_{\Omega_T}$. P denotes the projection operator from $L^2(\Omega(t))$ onto $H(\Omega(t))$. We assume in the present paper that $\Gamma(t)$ is smooth (at least uniformly of class C^3), and locally represented by the function $x_3 = \psi(x_1, x_2, t)$ (or $x_1 = \psi(x_2, x_3, t)$, or $x_2 = \psi(x_3, x_1, t)$). Now we are in the position to give the definitions of weak and strong solutions of (1.1), (1.2).

(u, ρ) is a weak solution of (1.1), (1.2) if u and ρ satisfy the following conditions:

(i) $u \in L^2(0, T; V(\Omega(t))) \cap L^\infty(0, T; H(\Omega(t)))$, $\rho \in L^\infty(\Omega_T)$, $\alpha \leq \rho \leq \beta$;

(ii) $$\int_0^T \{(\rho u, \partial_t \phi)_{\Omega(t)} + (\rho u \cdot \nabla \phi, u)_{\Omega(t)} - \mu((u, \phi))_{\Omega(t)} + (\rho f, \phi)_{\Omega(t)}\}\, dt =$$
$$= -(\rho_0 u_0, \phi(0))_{\Omega(0)} \qquad \forall \phi \in D(\Omega_T) \text{ with } \phi(T) = 0,$$
$$\partial_t \rho + u \cdot \nabla \rho = 0 \quad \text{in the sense of the distributions;}$$

(iii) $$\alpha |u(t)|^2_{\Omega(t)} + 2\mu \int_s^t \|u\|^2_{\Omega(\sigma)}\, d\sigma \leq \beta |u(s)|^2_{\Omega(s)} + 2 \int_s^t (\rho f, u)_{\Omega(\sigma)}\, d\sigma$$

holds for almost all $s > 0$, including $s = 0$, and all $t > s$.

(u, ρ) is a strong solution of (1.1), (1.2) if u and ρ satisfy the following conditions:

(i) $u \in L^2(0, T; H^2(\Omega(t))) \cap L^\infty(0, T; V(\Omega(t)))$, $\partial_t u \in L^2(\Omega_T)$,
$\rho \in L^\infty(\Omega_T)$, $\alpha \leq \rho \leq \beta$;

(ii) $P(\rho \partial_t u + \rho u \cdot \nabla u - \mu \Delta u - \rho f) = 0$ $a.e.$ in Ω_T

$$\partial_t \rho + u \cdot \nabla \rho = 0 \quad \text{in the sense of the distributions;}$$

$$(iii) \quad |\sqrt{\rho(t)}\, u(t)|^2_{\Omega(t)} + 2\mu \int_s^t \|u\|^2_{\Omega(\sigma)}\, d\sigma = |\sqrt{\rho(s)}\, u(s)|^2_{\Omega(s)} + 2\int_s^t (\rho f, u)_{\Omega(\sigma)}\, d\sigma$$

holds for almost all $s>0$, including $s=0$, and all $t>s$.

Our results are now given by the following theorems.

THEOREM 1. Let $u_0 \in H(\Omega(0)), f \in L^2(\Omega_T)$, and $\alpha \leqslant \rho_0 \leqslant \beta$. Then there exists a weak solution of (1.1), (1.2).

THEOREM 2. Let $u_0 \in V(\Omega(t))$, $f \in L^2(\Omega_T)$, and $\alpha \leqslant \rho_0 \leqslant \beta$. Then there exists a $0<\overline{T}\leqslant T$ such that there exists a strong solution of (1.1), (1.2) in $\Omega_{\overline{T}}$.

COROLLARY 3. The assumptions of theorem 2 hold. Further, we assume that $\|u_0\|_{\Omega(0)}$ and $|f|_{\Omega_T}$ are sufficiently small. Then there exists a strong solution of (1.1), (1.2) for every $T>0$.

THEOREM 4. The assumptions of theorem 1 hold, $\Omega(t)$ tends to a bounded domain Ω_0 as $t\to\infty$, and $|Pf|_{\Omega(t)} \leqslant ct^{-1/2}$. Then there exists a $T_0>0$ such that the weak solution of theorem 1 is a strong solution in (T_0, ∞) and u decays like $\|u\|^2_{\Omega(t)} \leqslant ct^{-1}$ where c is some positive constant.

THEOREM 5. The assumptions of Corollary 3 hold, $\Omega(t)$ tends to a bounded domain Ω_0 as $t\to\infty$, and $|Pf|_{\Omega(t)} \leqslant ct^{-1/2}$. Then there exists a strong solution of (1.1), (1.2) for every $T>0$ and u decays as $\|u\|^2_{\Omega(t)} \leqslant ct^{-1}$ where c is some positive constant.

3. PROOFS OF THEOREMS

First we prove theorem 1. We consider the following auxiliary problem. Let $\mathcal{F} = \{\phi | \phi \in L^2(0,T; H^2(\Omega(t))),\ \phi = 0 \text{ on } \Gamma_T \text{ with the natural norm}\}$, and $\mathcal{G} = \{\phi | \phi \in L^2(0,T; H^2(\Omega(t))), \partial_t \phi \in L^2(0,T; H^2(\Omega(t))), \phi = 0 \text{ on } \Gamma_T, \phi(T) = 0\}$. We consider on $\mathcal{G}$ the norm $\|\phi\|_{\mathcal{G}} = \|\phi\|_{\mathcal{F}} + \|\phi(0)\|_{\Omega(0)}$.

Find a $\pi_\varepsilon \in \mathcal{F}$ such that for all $\phi \in G$,

$$(3.1) \quad \int_0^T \{(\pi_\varepsilon, \Delta \partial_t \phi)_{\Omega(t)} + \varepsilon(\Delta\pi_\varepsilon, \Delta\phi)_{\Omega(t)} - (\tilde{\bar{u}} \cdot \nabla \pi_\varepsilon, \Delta\phi)_{\Omega(t)} - k(\pi_\varepsilon, \Delta\phi)_{\Omega(t)}\}\, dt =$$

$$= \int_0^T e^{-kt}(w, \Delta\phi)_{\Omega(t)}\, dt - ((\rho_0 - q(0)), \Delta\phi\}_{\Omega(0)};$$

here $w = -\partial_t q + \varepsilon \Delta q - \tilde{\bar{u}} \cdot \nabla q$; $\bar{u}$ and q are given functions, k is a positive constant, and $\tilde{\bar{u}}$ is a regularization of $\bar{u}$ through the use of a mollifier depending on a parameter λ, which is omitted for semplicity.

We denote by $E(\pi_\varepsilon, \phi)$ the left-hand side of (3.1), and by direct computation we have $E(\phi, \phi) \geqslant c_\varepsilon \|\phi\|^2_G$ (for suitable k); hence by [6, p.208], there exists a $\pi_\varepsilon \in \mathcal{F}$ such that (3.1) holds. Now $-\Delta$ is one to one and onto from $H^2(\Omega(t)) \cap H_0^1(\Omega(t))$ to $L^2(\Omega(t))$, then we have that π_ε satisfies

$$(3.2) \quad \partial_t \pi_\varepsilon - \varepsilon\Delta\pi_\varepsilon + \tilde{\bar{u}} \cdot \nabla\pi_\varepsilon + k\pi_\varepsilon = e^{-kt} w \qquad a.e. \text{ in } \Omega_T.$$

Now multiplying (3.2) by $\exp kt$ and setting $\rho_\varepsilon = \exp kt\, \pi_\varepsilon + q$, we have proved the existence of a solution of the system

$$\partial_t \rho_\varepsilon - \varepsilon \Delta \rho_\varepsilon + \tilde{\tilde{u}} \cdot \nabla \rho_\varepsilon = 0 \quad \text{in } \Omega_T, \qquad \rho_\varepsilon = q \quad \text{on } \Gamma_T. \tag{3.3}$$

Further, it is a routine matter to prove $\alpha \leqslant \rho_\varepsilon \leqslant \beta$.

By using the same method we can prove the existence of a solution u_ε of

$$P(\rho_\varepsilon \partial_t u_\varepsilon - \mu \Delta u_\varepsilon + \rho_\varepsilon \tilde{\tilde{u}} \cdot \nabla \bar{u} - \rho_\varepsilon f - k(1-\rho_\varepsilon)(\bar{u} - u_\varepsilon)) = 0. \tag{3.4}$$

Now combining fixed point arguments and *a priori* estimates proved in [3], passing to limit $\varepsilon \to 0$, and after $\lambda \to 0$, we have the existence of a weak solution of (1.1), (1.2).

To prove theorem 2, we consider the approximating system (3.3), (3.4) in which there is not the regularization of $\bar{u}$, in other words we have the terms $\bar{u} \cdot \nabla \rho_\varepsilon$ and $\rho_\varepsilon \bar{u} \cdot \nabla \bar{u}$ instead of $\tilde{\tilde{u}} \cdot \nabla \rho_\varepsilon$ and $\rho_\varepsilon \tilde{\tilde{u}} \cdot \nabla \bar{u}$, respectively. Using fixed point arguments, and then passing to limit $\varepsilon \to 0$, we prove the existence, locally in time, of a strong solution of (1.1), (1.2).

The proof of corollary 3 consists in finding a suitable bound for the data such that fixed point arguments of theorem 2 hold true for every $T > 0$. Now combining the estimates of theorem 2 and the results in [4], we can prove theorems 4, 5.

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