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On Cauchy's stress theorem

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Meccanica dei continui. — *On Cauchy's stress theorem.* Nota di MIROSLAV ŠILHAVÝ, presentata (*) dal Socio C. TRUESDELL.

ABSTRACT. — In this work a new proof of the theorem of Cauchy on the existence of the stress tensor is given which does not use the tetrahedron argument.

KEY WORDS: Stress tensor; Continuous body; Balance equations.

RIASSUNTO. — *Teorema di Cauchy sull'esistenza del tensore degli sforzi.* In questo lavoro viene data una nuova dimostrazione del teorema di Cauchy sull'esistenza del tensore degli sforzi che non fa uso dell'argomento del tetraedro.

1. INTRODUCTION

The classical theorem of Cauchy, describing the contact forces in a continuous body in terms of the stress tensor, has been reconsidered by several authors [1-6] in the last three decades. These works have gradually removed various conceptual or technical assumptions made by Cauchy. The proofs in [1-4] use Cauchy's original tetrahedron argument or refinements of it. In [5, 6] I gave proofs using a different kind of argument based on the «slicing of the body». This method permits one to establish the existence of the stress tensor under assumptions more general than those considered hitherto. For instance, in [6] the surface tractions are required to be well defined only for «almost every surface» and the class of all such contact forces is shown to be isomorphic with the space of all integrable vector fields with integrable distributional divergences. This result gives not only the existence of the stress tensor, but also the validity of the divergence theorem without extra hypotheses of smoothnesss on the stress tensor.

In this note I give another proof of the stress theorem which does not use the tetrahedron argument. The proof leads to an explicit, coordinate-free formula for the stress tensor in terms of the surface tractions (see eq. (4) below). I do not aim at the greatest generality here; rather I wish to give a relatively simple proof which uses ideas somewhat different from the traditional ones. My assumptions are very close to Cauchy's. Notably, I adopt Cauchy's postulate, *i.e.*, I assume that the tractions associated with a surface at a point depend upon that surface only through its normal.

2. ASSUMPTIONS AND STATEMENT OF THE THEOREM

The body is identified with an open set $\mathbf{R}$ in a three-dimensional Euclidean space. We denote by $B(\boldsymbol{x}, r)$ the open ball of radius r centered at $\boldsymbol{x}$ and by $\boldsymbol{e} = \boldsymbol{e}(\boldsymbol{y})$ the unit

(*) Nella seduta del 10 marzo 1990.

outward normal to the boundary $\partial B(x, r)$ given by

$$(1) \qquad e(y) = (y - x)/r$$

for every $y \in \partial B(x, r)$.

The density of the surface traction on an oriented surface with normal n, $|n| = 1$, is given by a vector-valued function $t = t(x, n)$, $x \in \mathbf{R}$, $|n| = 1$ and I assume that for every fixed n the function $t(x, n)$ is continuous in x and that the function $t(y, e(y))$ is integrable on every sphere contained in $\mathbf{R}$. The volume density of the external body force is given by a continuous vector valued function $b = b(x)$, $x \in \mathbf{R}$. The integral equation of balance of forces

$$(2) \qquad \int_{\partial P} t(x, n)\, dA + \int_P b\, dV = 0$$

is postulated to hold for every solid spherical cap P, *i.e.*, for every set P of the form

$$(3) \qquad B(x, r) \cap \{y : (y - x) \cdot n < s\}$$

for some x, unit vector n, and real number s. In (2), ∂P is the boundary of P, $n = n(x)$ is the unit outward normal to ∂P, dA is the element of area of ∂P, and dV the element of volume of P.

Under the above assumptions, the following version of Cauchy's stress theorem holds.

THEOREM. *For every $x \in \mathbf{R}$ the limit*

$$(4) \qquad T(x) = \lim_{r \to 0} \left(\frac{4}{3}\pi r^3\right)^{-1} \Big/ \int_{\partial B(x, r)} t(y, e) \otimes (y - x)\, dA(y)$$

exists and satisfies

$$(5) \qquad t(x, n) = T(x)\, n$$

for every unit vector n.

In (4), $e = e(y)$ is given by (1), the tensor product of two vectors is given in indices by $(a \otimes b)_{ij} = a_i b_j$ and $T(x)n$ denotes the vector with components $(T(x)\, n)_i = T_{ij}(x)\, n_j$.

3. PROOF

Let $B(x, r)$ be a ball such that its closure is contained in $\mathbf{R}$ and let n be any unit vector. Denote, for $s \in [-r, r]$, by $P(s)$ the solid spherical cap (3). Its boundary is given by $\partial P(s) = S(s) \cup T(s)$, where $S(s)$ is the portion of $\partial P(s)$ contained in $\partial B(x, r)$ and $T(s)$ is the open circular disc forming the complement of $T(s)$ in $\partial P(s)$. The equation of balance of forces (2) applied to $P(s)$ gives

$$(6) \qquad \int_{T(s)} t(y, n)\, dA + \int_{S(s)} t(y, e)\, dA + \int_{P(s)} b\, dV = 0$$

for every $s \in (-r, r]$. Here we have taken into account the fact that the normal to $P(s)$ is equal to $\boldsymbol{n}$ on $T(s)$ and to $\boldsymbol{e}(\boldsymbol{y})$ on $S(s)$. Rearrange (6):

$$\int_{T(s)} \boldsymbol{t}(\boldsymbol{y},\boldsymbol{n})\,dA = -\int_{S(s)} \boldsymbol{t}(\boldsymbol{y},\boldsymbol{e})\,dA - \int_{P(s)} \boldsymbol{b}\,dV$$

and integrate with respect to s from $-r$ to r:

$$\int_{-r}^{r}\left(\int_{T(s)} \boldsymbol{t}(\boldsymbol{y},\boldsymbol{n})\,dA\right)ds = -\int_{-r}^{r}\left(\int_{S(s)} \boldsymbol{t}(\boldsymbol{y},\boldsymbol{e})\,dA + \int_{P(s)} \boldsymbol{b}\,dV\right)ds. \tag{7}$$

By Fubini's theorem

$$\int_{-r}^{r}\left(\int_{T(s)} \boldsymbol{t}(\boldsymbol{y},\boldsymbol{n})\,dA\right)ds = \int_{B(\boldsymbol{x},r)} \boldsymbol{t}(\boldsymbol{y},\boldsymbol{n})\,dV(\boldsymbol{y}). \tag{8}$$

I now claim that

$$\begin{aligned} -\int_{-r}^{r}\left(\int_{S(s)} \boldsymbol{t}(\boldsymbol{y},\boldsymbol{e})\,dA + \int_{P(s)} \boldsymbol{b}\,dV\right)ds = \\ = \left[\int_{\partial B(\boldsymbol{x},r)} \boldsymbol{t}(\boldsymbol{y},\boldsymbol{e}) \otimes (\boldsymbol{y}-\boldsymbol{x})\,dA(\boldsymbol{y}) + \int_{B(\boldsymbol{x},r)} \boldsymbol{b} \otimes (\boldsymbol{y}-\boldsymbol{x})\,dV\right]\cdot \boldsymbol{n}. \end{aligned} \tag{9}$$

Indeed, consider the integral

$$I = \int_{\tilde{B}} \boldsymbol{b}(\boldsymbol{y})\,dV(\boldsymbol{y})\,ds$$

over the set $\tilde{B} = \{(\boldsymbol{y}, s) : \boldsymbol{y} \in B(\boldsymbol{x}, r), s \in [-r, r], s \geqslant (\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}\}$. Fubini's theorem tells us that

$$I = \int_{-r}^{r}\left(\int_{\tilde{B}_{*,s}} \boldsymbol{b}\,dV\right)ds = \int_{B(\boldsymbol{x},r)}\left(\int_{\tilde{B}_{\boldsymbol{y},*}} \boldsymbol{b}\,ds\right)dV(\boldsymbol{y}) \tag{10}$$

where $\tilde{B}_{*,s}$ and $\tilde{B}_{\boldsymbol{y},*}$ are, respectively, the horizontal and the vertical sections of $\tilde{B}$, defined by $\tilde{B}_{*,s} = \{\boldsymbol{y} : (\boldsymbol{y}, s) \in \tilde{B}\}$ and $\tilde{B}_{\boldsymbol{y},*} = \{s : (\boldsymbol{y}, s) \in \tilde{B}\}$. Using these definitions, we see easily that $\tilde{B}_{*,s} = P(s)$ and that $\tilde{B}_{\boldsymbol{y},*}$ is the closed interval $[(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}, r]$. Equation $(10)_2$ then reduces to

$$\int_{-r}^{r}\left(\int_{P(s)} \boldsymbol{b}\,dV\right)ds = \int_{B(\boldsymbol{x},r)}\left(\int_{(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}}^{r} \boldsymbol{b}(\boldsymbol{y})\,ds\right)dV(\boldsymbol{y}) = \int_{B(\boldsymbol{x},r)} \boldsymbol{b}(\boldsymbol{y})(r-(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n})\,dV. \tag{11}$$

A similar argument provides

$$\int_{-r}^{r}\left(\int_{S(s)} \boldsymbol{t}(\boldsymbol{y},\boldsymbol{e})\,dA\right)ds = \int_{\partial B(\boldsymbol{x},r)} \boldsymbol{t}(\boldsymbol{y},\boldsymbol{e})(r-(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n})\,dA \tag{12}$$

and hence

$$\int_{-r}^{r}\left(\int_{S(s)} t(y,e)\,dA+\int_{P(s)} b\,dV\right)ds=-\int_{\partial B(x,r)} t(y,e)(y-x)\cdot n\,dA$$
$$-\int_{B(x,r)} b(y)(y-x)\cdot n\,dV+r\left(\int_{\partial B(x,r)} t(y,e)\,dA+\int_{B(x,r)} b\,dV\right).$$

Noticing that the last term in the last expression vanishes because of the equation of balance of forces (2) and using elementary properties of the tensor product, we arrive at (9). Equations (7), (8) and (9) give the following important formula:

$$\int_{B(x,r)} t(y,n)\,dV=\left[\int_{\partial B(x,r)} t(y,e)\otimes(y-x)\,dA+\int_{B(x,r)} b\otimes(y-x)\,dV\right]\cdot n. \tag{13}$$

We now divide (13) by $(4/3)\pi r^3$ and let $r\to 0$. Because of the assumption that $t(x, n)$ is continuous, the limit of the left-hand side of the equation (13) after division exists and equals $t(x, n)$. In the limit contribution of the volume-integral on the right-hand side of equation (13) after division, being generally the value of the integrand at x, is 0 in the present case. This fact and the existence of the limit of the left-hand side implies that the limit

$$\lim_{r\to 0}\left(\frac{4}{3}\pi r^3\right)^{-1}\int_{\partial B(x,r)} t(y,e)\otimes(y-x)\,dA(y)\cdot n$$

exists and equals the limit of the left-hand side, *i.e.*,

$$t(x,n)=\lim_{r\to 0}\left(\frac{4}{3}\pi r^3\right)^{-1}\int_{\partial B(x,r)} t(y,e)\otimes(y-x)\,dA(y)\cdot n.$$

Since n is an arbitrary unit vector, this gives the existence of the limit (4) and the formula (5). The proof is complete.

4. Symmetry of the stress tensor

I shall now show that the formula (4) gives a very natural proof of the symmetry of the stress tensor $T(x)$. Accordingly, suppose now additionally that also the angular momentum is balanced in the sense that

$$\int_{\partial B(x,r)} (y-x)\times t(y,e)\,dA+\int_{B(x,r)} (y-x)\times b(y)\,dV=0 \tag{14}$$

for every $B(x,r)$ whose closure is contained in $\mathbf{R}$. Equation (14) amounts to saying that the skew part of the tensor

$$\int_{\partial B(x,r)} (y-x)\otimes t(y,e)\,dA+\int_{B(x,r)} (y-x)\otimes b(y)\,dV \tag{15}$$

vanishes. Dividing (15) by $(4/3)\pi r^3$ and letting $r\to 0$, we deduce that the skew part of the corresponding limit (if it exists) vanishes. But the limit of the first term of the expression (15) after division is the transpose of $\boldsymbol{T}(\boldsymbol{x})$ by (4) while the limit of the second term in (15) after division is 0 as in the proof of the Theorem. Hence the skew part of the transpose of $\boldsymbol{T}(\boldsymbol{x})$ is 0 and this is the symmetry of $\boldsymbol{T}(\boldsymbol{x})$.

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