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On the mechanical behaviour of laminated curved beams: a simple model which takes into account the warping effects

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Meccanica. — *On the mechanical behaviour of laminated curved beams: a simple model which takes into account the warping effects.* Nota di Luigi Ascione e Fernando Fraternali, presentata (*) dal Socio E. Giangreco.

Abstract. — A mechanical one-dimensional model which describes the dynamical behaviour of laminated curved beams is formulated. It is assumed that each lamina can be regarded as a Timoshenko's beam and that the rotations of the cross sections can differ from one lamina to another. The relative displacements at the interfaces of adjacent laminae are assumed to be zero. Consequently the model includes a shear deformability, due to the warping of the cross beam section consequent to the variability of the laminae rotations, but it is not able to describe the phenomena of the interfacial slip and of the delamination. By means of the principle of virtual work a variational formulation of the motion equations is carried out, which is useful in view of a successive numerical approach. The constitutive assumption at the basis of the analysis carried out is that of an orthotropic linear elastic behaviour in the single lamina.

Key words: Composite materials; Curved beams; Dynamics; Constitutive relations.

Riassunto. — *Sul comportamento meccanico di travi curve laminate: un modello semplice che tiene conto degli effetti di ingobbamento.* Si formula un modello meccanico monodimensionale, che descrive il comportamento dinamico di travi curve laminate. Si suppone che ciascuna lamina si comporti come una trave di Timoshenko con rotazioni delle sezioni che possono variare da lamina a lamina. Gli spostamenti relativi all'interfaccia di lamine adiacenti si suppongono nulli. Conseguentemente il modello include una deformabilità tagliante, consentita dall'ingobbamento dovuto alla variabilità delle suddette rotazioni, ma non è in grado di descrivere i fenomeni di scorrimento interfacciale e di delaminazione. Attraverso il principio dei lavori virtuali si sviluppa una formulazione variazionale delle equazioni del moto in vista di un successivo approccio numerico agli elementi finiti. L'ipotesi costitutiva a base dell'analisi svolta è quella di comportamento elastico lineare ortotropo nell'ambito della singola lamina.

1. Introduction

In recent years the interest of the scientists for fibrous composite materials has progressively grown due to the even wider use of these materials in commercial applications.

The laminar composites – in form of laminated beams, plates and shells – represent the tipology of wider use. In particular, laminated curved beams are used extensively as structural members or stiffners of thin shells.

The formulation of a one-dimensional model for laminates curved beams is the object of this paper.

A crucial point in formulating such a model is represented by the shear deformability, which is modeled in the literature by means of one of the following approaches.

A first one is founded upon the hypotheses of the classical Timoshenko's theory and requires the introduction of a suitable shear correction factor (see for example [1-4]).

(*) Nella seduta del 10 marzo 1990.

A second approach consists in assuming a particular shape of the warping displacements exhibited by the points of the cross section (see for example [5]).

A third approach has been recently proposed by Yuan and Miller [6] for a laminated, rectilinear beam. This approach regards the laminae as Timoshenko beams perfectly bonded at the interfaces. As a result of different rotations of the laminae, a piece-wise linear shape of the axial displacement field is obtainable.

In this paper we present an extension of the third approach in the case of plane curved beams.

Consequently, the kinematical model we introduce depends on $n+2$ generalized displacements which are the rotations of the n laminae, the radial displacement and the mean axial displacement of the cross section.

Our main goal in this paper is to obtain a variational formulation of the motion equations, useful to get numerical solutions.

The basic assumption we set about to define the generalized constitutive equations of the one-dimensional model is that the single lamina is composed of an orthotropic linear elastic material.

2. Formulation of the dynamical problem of a single lamina

First, we should consider the dynamical equilibrium problem of a single lamina isolated by the laminated beam (fig. 1).

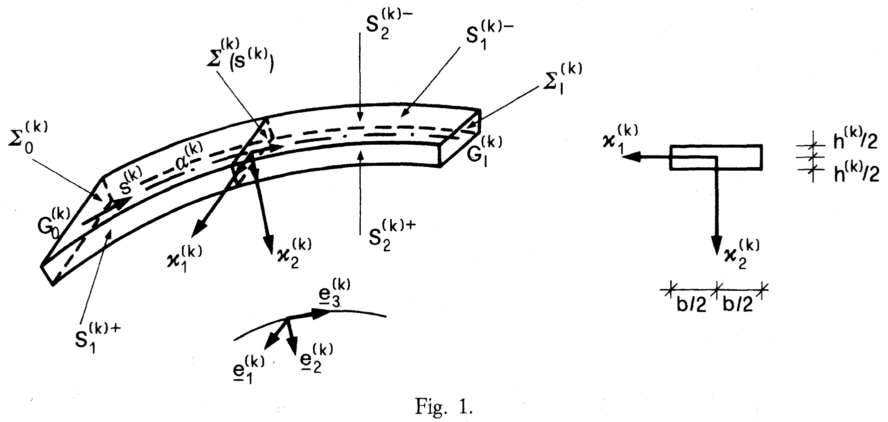

Fig. 1.

In the following we will write $L^{(k)}$ to denote the lamina under consideration, $\partial L^{(k)}$ for its boundary and $\alpha^{(k)}$ for the locus of centroids of its cross sections. We suppose that $\alpha^{(k)}$ is a plane regular curve and that the lamina has a rectangular cross section and is homogeneous. We will also use the notation $\Sigma^{(k)}(s^{(k)})$ for the cross section corresponding to the value $s^{(k)}$ of the curvilinear coordinate along $\alpha^{(k)}$ and will indicate with $\{\underline{e}_1^{(k)}, \underline{e}_2^{(k)}, \underline{e}_3^{(k)}\}$ the local frame along $\alpha^{(k)}$ defined in the order by the unit binormal vector, the unit normal (principal) vector and the unit tangent vector (Frénet frame) (fig. 1).

It is convenient to express $\partial L^{(k)}$ as:

(2.1) $$\partial L^{(k)} = S_1^{(k)+} \cup S_1^{(k)-} \cup S_2^{(k)+} \cup S_2^{(k)-} \cup \Sigma_0^{(k)} \cup \Sigma_l^{(k)},$$

being $S_1^{(k)+}$, $S_1^{(k)-}$, $S_2^{(k)+}$, $S_2^{(k)-}$ respectively the surfaces with unit normal $\underline{e}_1^{(k)}$, $-\underline{e}_1^{(k)}$, $\underline{e}_2^{(k)}$, $-\underline{e}_2^{(k)}$ and $\Sigma_0^{(k)}$, $\Sigma_l^{(k)}$ the two bases of the lamina (fig. 1).

2.1. Kinematical assumptions

We assume that the lamina can be studied as a plane beam, whose cross section remains plane during the motion but not necessarily perpendicular to the deformed centerline (Timoshenko's hypothesis).

Referring to the fig. 1, the components of the displacement field $\underline{u}^{(k)}$ with respect to the frame $\{\underline{e}_1^{(k)}, \underline{e}_2^{(k)}, \underline{e}_3^{(k)}\}$ can therefore be expressed in the form:

(2.2a) $$u_1^{(k)}(x_1^{(k)}, x_2^{(k)}, s^{(k)}, t) = 0,$$

(2.2b) $$u_2^{(k)}(x_1^{(k)}, x_2^{(k)}, s^{(k)}, t) = v_2^{(k)}(s^{(k)}, t),$$

(2.2c) $$u_3^{(k)}(x_1^{(k)}, x_2^{(k)}, s^{(k)}, t) = v_3^{(k)}(s^{(k)}, t) + x_2^{(k)} \phi_1^{(k)}(s^{(k)}, t),$$

in which $v_2^{(k)}$ and $v_3^{(k)}$ are respectively the radial and the tangential displacements of the centerline and $\varphi_1^{(k)}$ is the rotation (about the $x_1^{(k)}$ axis) of the cross section.

By deducing the displacement gradient from the components $u_1^{(k)}$, $u_2^{(k)}$, $u_3^{(k)}$ [7], we can write the strain-displacement relations in the form:

(2.3a) $$\varepsilon_{11}^{(k)} = \partial u_1^{(k)}/\partial x_1^{(k)},$$

(2.3b) $$\varepsilon_{22}^{(k)} = \partial u_2^{(k)}/\partial x_2^{(k)},$$

(2.3c) $$\varepsilon_{33}^{(k)} = \rho^{(k)}(\rho^{(k)} - x_2^{(k)})^{-1}(\partial u_3^{(k)}/\partial s^{(k)} - u_2^{(k)}/\rho^{(k)}),$$

(2.3d) $$\varepsilon_{12}^{(k)} = 1/2\,(\partial u_1^{(k)}/\partial x_2^{(k)} + \partial u_2^{(k)}/\partial x_1^{(k)}),$$

(2.3e) $$\varepsilon_{31}^{(k)} = 1/2\,[\partial u_3^{(k)}/\partial x_1^{(k)} + \rho^{(k)}(\rho^{(k)} - x_2^{(k)})^{-1}\partial u_1^{(k)}/\partial s^{(k)}],$$

(2.3f) $$\varepsilon_{32}^{(k)} = 1/2\,[\partial u_3^{(k)}/\partial x_2^{(k)} + \rho^{(k)}(\rho^{(k)} - x_2^{(k)})^{-1}(\partial u_2^{(k)}/\partial s^{(k)} + u_3^{(k)}/\rho^{(k)})],$$

where $\rho^{(k)}$ is the radius of curvature of $\alpha^{(k)}$.

From the eqs. (2.2) and (2.3) it follows that the only nonzero strain components are:

(2.4a) $$\varepsilon_{33}^{(k)} = \rho^{(k)}(\rho^{(k)} - x_2^{(k)})^{-1}(\partial v_3^{(k)}/\partial s^{(k)} - v_2^{(k)}/\rho^{(k)} + x_2^{(k)}\,\partial\phi_1^{(k)}/\partial s^{(k)}),$$

(2.4b) $$\varepsilon_{32}^{(k)} = 1/2\,\rho^{(k)}(\rho^{(k)} - x_2^{(k)})^{-1}(\partial v_2^{(k)}/\partial s^{(k)} + v_3^{(k)}/\rho^{(k)} + \phi_1^{(k)}).$$

2.2. Equations of motion

Using the symbols of the figs. 1 and 2, we write the equations of motion of the

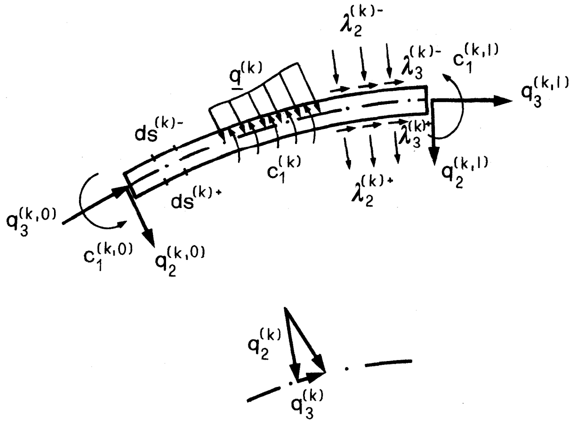

Fig. 2.

lamina in a variational form by means of the principle of virtual work:

$$(2.5)\qquad \int_{\vec{\alpha}^{(k)}} \int_{\Sigma^{(k)}(s^{(k)})} (\sigma_{33}^{(k)}\,\delta\varepsilon_{33}^{(k)} + 2\sigma_{32}^{(k)}\,\delta\varepsilon^{(k)})\,(\rho^{(k)} - x_2^{(k)})/\rho^{(k)}\,dx_1^{(k)}\,dx_2^{(k)}\,ds^{(k)} =$$

$$= \int_{\vec{\alpha}^{(k)}} (q_2^{(k)}\,\delta v_2^{(k)} + q_3^{(k)}\,\delta v_3^{(k)} + c_1(k)\,\delta\phi_1^{(k)})\,ds^{(k)} +$$

$$+ \int_{\vec{\alpha}^{(k)+}} [\lambda_2^{(k)+}\delta v_2^{(k)} + \lambda_3^{(k)+}\,(\delta v_3^{(k)} + h^{(k)}\,\delta\phi_1^{(k)}/2)]\,ds^{(k)+} +$$

$$+ \int_{\vec{\alpha}^{(k)-}} [\lambda_2^{(k)-}\,\delta v_2^{(k)} + \lambda_3^{(k)-}\,(\delta v_3^{(k)} - h^{(k)}\delta\phi_1^{(k)}/2)]\,ds^{(k)-} +$$

$$+ \int_{\vec{\alpha}^{(k)}} \int_{\Sigma^{(k)}(s^{(k)})} [\mu^{(k)}\,\ddot{v}_2^{(k)}\,\delta v_2^{(k)} + \mu^{(k)}\,(\ddot{v}_3^{(k)} + x_2^{(k)}\,\ddot{\phi}_1^{(k)})\,(\delta v_3^{(k)} + x_2^{(k)}\,\delta\phi_1^{(k)})]\,(\rho^{(k)} - x_2^{(k)})/\rho^{(k)}\,dx_1^{(k)}\,dx_2^{(k)}\,ds^{(k)} +$$

$$+ q_2^{(k,o)}\,\delta v_2^{(k,o)} + q_3^{(k,o)}\,\delta v_3^{(k,o)} + c_1^{(k,o)}\,\delta\phi_1(k,o) + q_2^{(k,l)}\,\delta v_2^{(k,l)} + q_3^{(k,l)}\,\delta v_3^{(k,l)} + c_1^{(k,l)}\,\delta\phi_1^{(k,l)}$$

which must hold for each virtual displacement $\delta v_2^{(k)}$, $\delta v_3^{(k)}$, $\delta\varphi_1^{(k)}$.

The first member of the eq. (2.5) represents the virtual work of the internal forces (work of the stresses $\sigma_{ij}^{(k)}$ for the virtual strain components $\delta\varepsilon_{ij}^{(k)}$, related to $\delta v_2^{(k)}$, $\delta v_3^{(k)}$, $\delta\varphi_1^{(k)}$ through the relations (2.4)).

The first term of the second member represents the virtual work of the external loads $q_2^{(k)}$, $q_3^{(k)}$, $c_1^{(k)}$, which are defined per unit of length of the centerline and derive from the body forces and the surface forces on $S_1^{(k)+}$ and $S_1^{(k)-}$ (fig. 2).

The second and the third terms represent the virtual works of the loads $\lambda_i^{(k)+}$ and $\lambda_i^{(k)-}$ ($i = 2, 3$), which act respectively on the surfaces $S_2^{(k)+}$ and $S_2^{(k)-}$ and are defined per unit of length of the two lines $\alpha^{(k)+}$ and $\alpha^{(k)-}$ intersections of these surfaces with the plane of the lamina (fig. 2).

The loads $\lambda_i^{(k)+}$ and $\lambda_i^{(k)-}$ derive respectively from the surfaces forces on $S_2^{(k)+}$ and $S_2^{(k)-}$, which are due, in particular, to the mutual actions between this and the other laminae.

The fourth term of the second member represents the virtual work of the inertial forces $-\mu^{(k)}\ddot{u}_2^{(k)}$ and $-\mu^{(k)}\ddot{u}_3^{(k)}$.

Finally the last terms of the second member represent the virtual works of the forces acting on the ends of the lamina (fig. 2).

By the use of the eqs. (2.4) and of the following geometrical relations:

$$ds^{(k)+} = (1 - h^{(k)}/2\rho^{(k)})\, ds^{(k)}, \tag{2.6a}$$

$$ds^{(k)-} = (1 + h^{(k)}/2\rho^{(k)})\, ds^{(k)}, \tag{2.6b}$$

we may transform the principle (2.5) into:

$$\begin{aligned}(2.7)\quad &\int_{\vec{\alpha}^{(k)}} (T_2^{(k)}\,\delta\gamma_2^{(k)} + N^{(k)}\,\delta\varepsilon^{(k)} + M_1^{(k)}\,\delta\chi_1^{(k)})\, ds^{(k)} = \\ &+ \int_{\vec{\alpha}^{(k)}} (q_2^{(k)}\,\delta v_2^{(k)} + q_3^{(k)}\,\delta v_3^{(k)} + c_1^{(k)}\,\delta\phi_1^{(k)})\, ds^{(k)} + \\ &+ \int_{\vec{\alpha}^{(k)}} (\lambda_2^{(k)+}\,\delta v_2^{(k)} + \lambda_3^{(k)+}\,\delta v_3^{(k)} + h^{(k)}\,\lambda_3^{(k)+}\,\delta\phi_1^{(k)}/2)\,(1 - h^{(k)}/2\rho^{(k)})\, ds^{(k)} + \\ &+ \int_{\vec{\alpha}^{(k)}} (\lambda_2^{(k)-}\,\delta v_2^{(k)} + \lambda_3^{(k)-}\,\delta v_3^{(k)} - h^{(k)}\,\lambda_3^{(k)-}\delta\phi_1^{(k)}/2)\,(1 + h^{(k)}/2\rho^{(k)})\, ds^{(k)} - \\ &- \int_{\vec{\alpha}^{(k)}} (m^{(k)}\,\ddot{v}_2^{(k)}\,\delta v_2^{(k)} + m^{(k)}\,\ddot{v}_3^{(k)}\,\delta v_3^{(k)} + I_m^{(k)}\,\ddot{v}_3^{(k)}\,\delta\phi_1^{(k)}/\rho^{(k)} + I_m^{(k)}\,\ddot{\phi}_1^{(k)}\delta v_3^{(k)}/\rho^{(k)} + I_m^{(k)}\,\ddot{\phi}_1^{(k)}\delta\phi_1^{(k)})\, ds^{(k)} + \\ &\qquad + q_2^{(k,0)}\,\delta v_2^{(k,0)} + q_3^{(k,0)}\,\delta v_3^{(k,0)} + c_1^{(k,0)}\,\delta\phi_1^{(k,0)} + q_2^{(k,l)}\,\delta v_2^{(k,l)} + q_3^{(k,l)}\,\delta v_3^{(k,l)} + c_1^{(k,l)}\,\delta\phi_1^{(k,l)},\end{aligned}$$

where we have posed:

$$T_2^{(k)} = \int_{\Sigma^{(k)}} \sigma_{32}^{(k)}\, dx_1^{(k)}\, dx_2^{(k)}, \tag{2.8a}$$

$$N^{(k)} = \int_{\Sigma^{(k)}} \sigma_{33}^{(k)}\, dx_1^{(k)}\, dx_2^{(k)}, \tag{2.8b}$$

$$M_1^{(k)} = \int_{\Sigma^{(k)}} \sigma_{33}^{(k)}\, x_2^{(k)}\, dx_1^{(k)}\, dx_2^{(k)}; \tag{2.8c}$$

$$\gamma_2^{(k)} = \partial v_2^{(k)}/\partial s^{(k)} + v_3^{(k)}/\rho^{(k)} + \phi_1^{(k)}, \tag{2.9a}$$

$$\varepsilon^{(k)} = \partial v_3^{(k)}/\partial s^{(k)} - v_2^{(k)}/\rho^{(k)}, \tag{2.9b}$$

$$\chi_1^{(k)} = \partial\phi_1^{(k)}/\partial s^{(k)}; \tag{2.9c}$$

$$m^{(k)} = \int_{\Sigma^{(k)}} \mu^{(k)}\, dx_1^{(k)}\, dx_2^{(k)} = \mu^{(k)}\, A^{(k)}, \tag{2.10a}$$

$$I_m^{(k)} = \int_{\Sigma^{(k)}} \mu^{(k)}\,(x_2^{(k)})^2\, dx_1^{(k)}\, dx_2^{(k)} = \mu^{(k)}\, I^{(k)}, \tag{2.10b}$$

$A^{(k)}$ being the area of the cross section and $I^{(k)}$ its moment of inertia about the $x_1^{(k)}$ axis.

The quantities defined by the eqs. (2.8) and (2.9) represent respectively the stress resultants and the generalized strains, $m^{(k)}$ is the mass of the cross section and $I_m^{(k)}$ is the mass moment of inertia of the cross section (about the $x_1^{(k)}$ axis).

Introducing now the numerical vectors:

$$(2.11a,b,c) \qquad \boldsymbol{u}^{(k)} = \begin{bmatrix} v_2^{(k)} \\ v_3^{(k)} \\ \phi_1^{(k)} \end{bmatrix}, \quad \boldsymbol{\varepsilon}^{(k)} = \begin{bmatrix} \gamma_2^{(k)} \\ \varepsilon^{(k)} \\ \chi_1^{(k)} \end{bmatrix}, \quad \boldsymbol{\sigma}^{(k)} = \begin{bmatrix} T_2^{(k)} \\ N^{(k)} \\ M_1^{(k)} \end{bmatrix},$$

$$(2.11d,e,f) \qquad \boldsymbol{q}^{(k)} = \begin{bmatrix} q_2^{(k)} \\ q_3^{(k)} \\ c_1^{(k)} \end{bmatrix}, \quad \boldsymbol{q}_0^{(k)} = \begin{bmatrix} q_2^{(k,0)} \\ q_3^{(k,0)} \\ c_1^{(k,0)} \end{bmatrix}, \quad \boldsymbol{q}_l^{(k)} = \begin{bmatrix} q_2^{(k,l)} \\ q_3^{(k,l)} \\ c_1^{(k,l)} \end{bmatrix},$$

$$(2.11g,h) \qquad \boldsymbol{\lambda}^{(k)+} = \begin{bmatrix} \lambda_2^{(k)+} \\ \lambda_3^{(k)+} \\ h^{(k)}\lambda_3^{(k)+}/2 \end{bmatrix}, \quad \boldsymbol{\lambda}^{(k)-} = \begin{bmatrix} \lambda_2^{(k)-} \\ \lambda_3^{(k)-} \\ -h^{(k)}\lambda_3^{(k)-}/2 \end{bmatrix},$$

we can restate the eq. (2.7) in the following more compact form:

$$(2.12) \qquad \int_{\vec{\alpha}^{(k)}} \delta\boldsymbol{\varepsilon}^{(k)^T} \boldsymbol{\sigma}^{(k)}\, ds^{(k)} = \int_{\vec{\alpha}^{(k)}} \delta\boldsymbol{u}^{(k)^T} \boldsymbol{q}^{(k)}\, ds^{(k)} +$$

$$+ \int_{\vec{\alpha}^{(k)}} \delta\boldsymbol{u}^{(k)^T} \boldsymbol{\lambda}^{(k)+} (1 - h^{(k)}/2\rho^{(k)})\, ds^{(k)} + \int_{\vec{\alpha}^{(k)}} \delta\boldsymbol{u}^{(k)^T} \boldsymbol{\lambda}^{(k)-} (1 + h^{(k)}/2\rho^{(k)})\, ds^{(k)} -$$

$$- \int_{\vec{\alpha}^{(k)}} \delta\boldsymbol{u}^{(k)^T} \boldsymbol{M}^{(k)} \ddot{\boldsymbol{u}}^{(k)}\, ds^{(k)} + \delta\boldsymbol{u}_0^{(k)^T} \boldsymbol{q}_0^{(k)} + \delta\boldsymbol{u}_l^{(k)^T} \boldsymbol{q}_l^{(k)}$$

where:

$$(2.13) \qquad \boldsymbol{M}^{(k)} = \begin{bmatrix} m^{(k)} & 0 & 0 \\ 0 & m^{(k)} & -I_m^{(k)}/\rho^{(k)} \\ 0 & -I_m^{(k)}/\rho^{(k)} & I_m^{(k)} \end{bmatrix}.$$

Finally, we observe that the generalized strain-displacement relations (2.9) can be written as follows:

$$(2.14) \qquad \boldsymbol{\varepsilon}^{(k)} = \boldsymbol{L}^{(k)} \boldsymbol{u}^{(k)}$$

where we have introduced the linear operator $\boldsymbol{L}^{(k)}$ defined by the equation:

$$(2.15) \qquad \boldsymbol{L}^{(k)} = \begin{bmatrix} \partial/\partial s^{(k)} & 1/\rho^{(k)} & 1 \\ -1/\rho^{(k)} & \partial/\partial s^{(k)} & 0 \\ 0 & 0 & \partial/\partial s^{(k)} \end{bmatrix}.$$

2.3. Constitutive relations

We assume that the lamina is composed of an orthotropic linearly-elastic material, whose principal directions coincide with the directions of the unit vectors $\underline{e}_1^{(k)}$, $\underline{e}_2^{(k)}$ and $\underline{e}_3^{(k)}$.

The stress-strain relations are therefore of the kind [8]:

$$(2.16)\qquad \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2Q_{66} \end{bmatrix}^{(k)} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{bmatrix}^{(k)}.$$

Then, by the use of the eqs. (2.4), we find that the relations of interest in the present case are:

$$(2.17a)\qquad \sigma_{33}^{(k)} = Q_{33}^{(k)}\,\varepsilon_{33}^{(k)} = Q_{33}^{(k)}\,\rho^{(k)}\,(\rho^{(k)} - x_2^{(k)})^{-1}(\varepsilon^{(k)} + x_2^{(k)}\,\chi_1^{(k)}),$$

$$(2.17b)\qquad \sigma_{32}^{(k)} = 2Q_{44}^{(k)}\,\varepsilon_{32}^{(k)} = Q_{44}^{(k)}\,\rho^{(k)}\,(\rho^{(k)} - x_2^{(k)})^{-1}\gamma_2^{(k)}.$$

By introducing now the eqs. (2.17) into the expressions of the stress resultants (2.8), we obtain the generalized constitutive relations as follows:

$$(2.18a)\qquad \boldsymbol{\sigma}^{(k)} = \mathbf{C}^{(k)}\,\boldsymbol{\varepsilon}^{(k)},$$

where:

$$(2.18b)\qquad \mathbf{C}^{(k)} = \begin{bmatrix} Q_{44}^{(k)}\,(A^{(k)} + J_\rho^{(k)}/\rho^{(k)^2}) & 0 & 0 \\ 0 & Q_{33}^{(k)}\,(A^{(k)} + J_\rho^{(k)}/\rho^{(k)^2}) & Q_{33}^{(k)}\,J_\rho^{(k)}/\rho^{(k)} \\ 0 & Q_{33}^{(k)}\,J_\rho^{(k)}/\rho^{(k)} & Q_{33}^{(k)}\,J_\rho^{(k)} \end{bmatrix},$$

being:

$$(2.18c)\qquad J_\rho^{(k)} = \int_{\Sigma^{(k)}} (x_2^{(k)})^2\,\rho^{(k)}\,(\rho^{(k)} - x_2^{(k)})^{-1}\,dx_1^{(k)}\,dx_2^{(k)} =$$

$$= \rho^{(k)^2}\,(b\rho^{(k)}\ln((\rho^{(k)} + h^{(k)}/2)/(\rho^{(k)} - h^{(k)}/2)) - bh^{(k)})$$

the «reduced» moment of inertia of the cross section.

3. Formulation of the dynamical problem of a laminated beam

Now we can study the dynamical equilibrium problem of a plane curved beam, which is considered to be made up of a number n of laminae of the kind previously described, each of which possesses different mechanical properties and thicknesses.

In the following we will indicate as «reference line» the locus α of the geometrical centroids of the beam cross sections and write s and $\Sigma(s)$ respectively for the curvilinear coordinate along α and for the generical cross section of the beam.

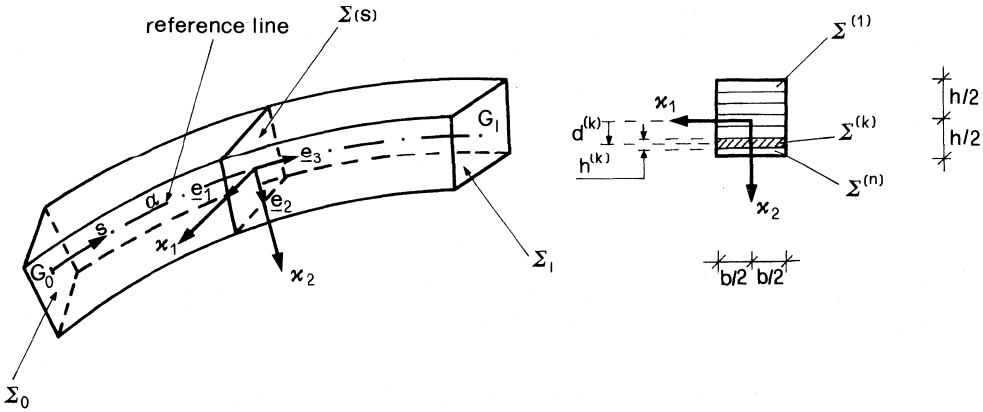

Fig. 3.

Referring to the fig. 3, it is easy to verify the following geometrical relations:

$$\rho^{(k)} = \rho - d^{(k)}, \tag{3.1a}$$

$$ds^{(k)} = \rho^{(k)}/\rho \, ds = (1 - d^{(k)}/\rho) \, ds, \qquad \forall k \in \{1, \ldots, n\}, \tag{3.1b}$$

where ρ is the radius of curvature of the curve α.

3.1. Kinematical assumptions

We suppose that the laminae are integrally bonded one to each other but that each lamina can rotate differently from the others.

By assuming the expressions (2.2) for the components of in each lamina, the constraint conditions previously defined translate into the following limitations:

$$v_2^{(k)} = v_2^{(k+1)}, \tag{3.2a}$$

$$v_3^{(k)} + h^{(k)} \phi_1^{(k)}/2 = v_3^{(k+1)} - h^{(k+1)} \phi_1^{(k+1)}/2, \qquad \forall k \in \{1, \ldots, n-1\}. \tag{3.2b}$$

Then, introducing the quantity:

$$\bar{v}_3 = (h)^{-1} \sum_{k=1}^{n} v_3^{(k)} h^{(k)}, \tag{3.3}$$

which represents the mean axial displacement of the generical cross section, it is easy to verify that the conditions (3.2) bring us to establish the following relation:

$$\boldsymbol{u}^{(k)} = \boldsymbol{A}^{(k)} \bar{\boldsymbol{u}} \tag{3.4}$$

where

$$\bar{\boldsymbol{u}} = [\bar{v}_2, \bar{v}_3, \phi_1^{(1)}, \ldots, \phi_1^{(n)}]^T; \tag{3.5}$$

$$A_{11}^{(k)} = 1, \quad A_{1,i}^{(k)} = 0 \quad \forall i \neq 1, \tag{3.6a}$$

$$A_{21}^{(k)} = 0, \; A_{22}^{(k)} = 1, \; A_{2,i+2}^{(k)} = h^{(i)} [d^{(i)}/h + \operatorname{sgn}(d^{(k)} - d^{(i)})/2] \qquad \forall i \in \{1, \ldots, n\}, \tag{3.6b}$$

$$(3.6c) \qquad A^{(k)}_{3,k+2} = 1\,, \quad A^{(k)}_{3,i} = 0 \quad \forall i \neq k+2\,, \quad \text{(no sum on } i\text{)},$$

being:

$$(3.7) \qquad \overline{v}_2 = v^{(1)}_2 = v^{(2)}_2 = \ldots = v^{(n)}_2\,.$$

3.2. Equations of motion

The equations of motions of the laminated beam, as well as those of the single lamina, can be expressed in a variational form by the use of the principle of virtual work.

By summing the contributions of each lamina and making use of the eqs. (3.1), the principle of virtual work for the laminated beam can be written as follows:

$$(3.8) \qquad \int_{\vec{\alpha}} \left[\sum_{k=1}^{n} (1 - d^{(k)}/\rho)\, \delta\boldsymbol{\varepsilon}^{(k)^T} \boldsymbol{\sigma}^{(k)} \right] ds = \int_{\vec{\alpha}} \delta\overline{\boldsymbol{u}}^T \overline{\boldsymbol{q}}\, ds - \int_{\vec{\alpha}} \delta\overline{\boldsymbol{u}}^T \overline{\boldsymbol{M}}\, \ddot{\overline{\boldsymbol{u}}}\, ds + \delta\overline{\boldsymbol{u}}_0^T \overline{\boldsymbol{q}}_0 + \delta\overline{\boldsymbol{u}}_l^T \overline{\boldsymbol{q}}_l\,,$$

where we have posed:

$$(3.9) \qquad \overline{\boldsymbol{q}} = \sum_{k=1}^{n} (1 - d^{(k)}/\rho)\, \boldsymbol{A}^{(k)^T} \boldsymbol{q}^{(k)} + (1 + h/2\rho)\, \boldsymbol{A}^{(1)^T} \boldsymbol{\lambda}^{(1)-} + (1 - h/2\rho)\, \boldsymbol{A}^{(n)^T} \boldsymbol{\lambda}^{(n)+}\,,$$

$$(3.10a,b) \qquad \overline{\boldsymbol{q}}_0 = \sum_{k=1}^{n} \boldsymbol{A}^{(k)^T} \boldsymbol{q}^{(k)}_0\,, \qquad \overline{\boldsymbol{q}}_l = \sum_{k=1}^{n} \boldsymbol{A}^{(k)^T} \boldsymbol{q}^{(k)}_l\,,$$

$$(3.11) \qquad \overline{\boldsymbol{M}} = \sum_{k=1}^{n} (1 - d^{(k)}/\rho)\, \boldsymbol{A}^{(k)^T} \boldsymbol{M}^{(k)} \boldsymbol{A}^{(k)}\,,$$

and we have taken into account that the virtual works of the mutual actions between adjacent laminae eliminate two by two. By the action and reaction law we have, infact, that $\lambda^{(k)+}_i = -\lambda^{(k)-}_i \ \forall k \in \{1, \ldots, n-1\}$.

We observe now that if we pose:

$$(3.12a) \qquad \overline{\boldsymbol{q}} = [\overline{q}_2, \overline{q}_3, \overline{c}^{(1)}_1, \ldots, \overline{c}^{(n)}_1]^T\,,$$

$$(3.12b) \qquad \overline{\boldsymbol{q}}_i = [\overline{q}^{(i)}_2, \overline{q}^{(i)}_3, \overline{c}^{(1,i)}_1, \ldots, \overline{c}^{(n,i)}_1]^T \qquad (i = 0, l)\,,$$

the eqs. (3.9) and (3.10), together with the eqs. (3.6), allow us to write:

$$(3.13a) \qquad \overline{q}_2 = \sum_{j=1}^{n} (1 - d^{(j)}/\rho)\, q^{(j)}_2 + (1 + h/2\rho)\, \lambda^{(1)-}_2 + (1 - h/2\rho)\, \lambda^{(n)-}_2\,,$$

$$(3.13b) \qquad \overline{q}_3 = \sum_{j=1}^{n} (1 - d^{(j)}/\rho)\, q^{(j)}_3 + (1 + h/2\rho)\, \lambda^{(1)-}_3 + (1 - h/2\rho)\, \lambda^{(n)-}_3\,,$$

$$(3.13c) \qquad \overline{c}^{(k)}_1 = (1 - d^{(k)}/\rho)\, c^{(k)}_1 + h^{(k)} d^{(k)}\, \overline{q}_3/h -$$
$$- h^{(k)}/2 \left[\sum_{j=1}^{n} (1 - d^{(j)}/\rho)\, q^{(j)}_3 \,\mathrm{sgn}\,(d^{(k)} - d^{(j)}) + (1 + h/2\rho)\, \lambda^{(1)-}_3 - (1 - h/2\rho)\, \lambda^{(n)+}_3 \right] -$$
$$- \delta_{k1} h^{(1)} (1 + h/2\rho)\, \lambda^{(1)-}_3/2 + \delta_{kn} h^{(n)} (1 - h/2\rho)\, \lambda^{(n)+}_3/2 \qquad \forall k \in \{1, \ldots, n\}\,, \quad \text{(no sum on } k\text{)},$$

$$\overline{q}_2^{(i)} = \sum_{j=1}^{n} q_2^{(j,i)}, \tag{3.14a}$$

$$\overline{q}_3^{(i)} = \sum_{j=1}^{n} q_3^{(j,i)}, \tag{3.14b}$$

$$\overline{c}_1^{(k,i)} = c_1^{(k,i)} + h^{(k)} d^{(k)} \overline{q}_3^{(i)}/h - (h^{(k)}/2) \sum_{j=1}^{n} q_3^{(j,i)} \operatorname{sgn}(d^{(k)} - d^{(j)}) \quad \forall k \in \{1, ..., n\}), \tag{3.14c}$$
$$(i = 0, l), \text{ (no sum on } k),$$

δ_{ij} being the Kronecker symbol.

In conclusion we observe that by substituing the eqs. (2.15) and (2.18) into the eq. (3.8), we obtain the following expression in terms of displacements of the principle of virtual work:

$$\int_{\vec{\alpha}} \delta\overline{\boldsymbol{u}}^T \overline{\boldsymbol{M}} \ddot{\overline{\boldsymbol{u}}}\, ds + \int_{\vec{\alpha}} \delta\overline{\boldsymbol{u}}^T \overline{\boldsymbol{K}}\, \overline{\boldsymbol{u}}\, ds = \int_{\vec{\alpha}} \delta\overline{\boldsymbol{u}}^T \overline{\boldsymbol{q}}\, ds + \delta\overline{\boldsymbol{u}}^T \overline{\boldsymbol{q}}_0 + \delta\overline{\boldsymbol{u}}_l^T \overline{\boldsymbol{q}}_l, \tag{3.15}$$

where we have introduced the linear operator $\overline{\boldsymbol{K}}$ formally defined by the equation:

$$\int_{\vec{\alpha}} \delta\overline{\boldsymbol{u}}^T \overline{\boldsymbol{K}}\, \overline{\boldsymbol{u}}\, ds = \int_{\vec{\alpha}} \sum_{k=1}^{n} (1 - d^{(k)}/\rho)\, \delta\overline{\boldsymbol{u}}^T \overline{\boldsymbol{A}}^{(k)^T} \boldsymbol{L}^{(k)^T} \boldsymbol{C}^{(k)} \boldsymbol{L}^{(k)} \boldsymbol{A}^{(k)} \overline{\boldsymbol{u}}\, ds. \tag{3.16}$$

4. Conclusions

We have presented a mechanical model which describes the dynamical behaviour of laminated curved beams in a simple but physically meaningful way.

We have modeled the shear deformability avoiding the definition of a shear correction factor or the assumption of a particular shape of the warping displacement field over the cross section.

The theory we have developed is particularly helpful in order to formulate a finite element approximation and infact we have just moved in this direction obtaining some numerical results which will be the subject of a future paper.

Furthermore, it is our intent to extend the approach here described in order to formulate a mechanical model for laminated curved beams loaded both in plane and out of the plane.

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