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Correction to my paper «Sull'analogo della formula di Selberg nei corpi di funzioni»

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Teoria dei numeri. — *Correction to my paper «Sull'analogo della formula di Selberg nei corpi di funzioni».* Nota (*) del Socio ENRICO BOMBIERI.

ABSTRACT. — This Note completes and corrects a preceding Lincean Note by introducing through a tauberian theorem an appropriate condition which removes a counter-example provided by Dr. Zhang.

KEY WORDS: Tauberian theorems; Abstract prime number theorems.

RIASSUNTO. — *Correzione alla mia Nota «Sull'analogo della formula di Selberg nei corpi di funzioni».* La presente Nota completa e corregge una Nota Lincea precedente introducendo, attraverso un teorema tauberiano, un'opportuna condizione che elimina il controesempio trovato dal Dott. Zhang.

It has been pointed out to me by Prof. H. Diamond and Dr. Wen Bin Zhang that the proof of Theorem 1 in the paper quoted in the title [1] is defective in the last part of the argument. The trouble arises from the fact that the left hand side of eq. (33) is not an approximation to its continuous form (28), so that the application of Wirsing's Lemma is not justified. In fact, Dr. Zhang makes the important observation that some other condition, other than (24) and $a_m \geqq 0$, is needed to infer that $a_m \sim 1$, as shown by the example $a_m = 2$ if m is odd, $a_m = 0$ if m is even.

The following tauberian theorem introduces an appropriate condition which removes the possibility of Dr. Zhang's example.

THEOREM 1. *Suppose that* $a_m \geqq 0$ *and*

$$ma_m + \sum_1^{m-1} a_i a_{m-i} = 2m + O(1) \tag{1}$$

$$ma_{2m} + \sum_1^{m-1} a_{2i} a_{2m-2i} = 2m + O(1)\,. \tag{2}$$

Then $a_m \sim 1$.

Let $a_m = 1 + r_m$. As in our earlier paper, the positivity of a_m and (1) imply

$$mr_m + \sum_{k<m} r_k r_{m-k} = O(1) \tag{3}$$

$$|r_m| \leqq 1 + O(1/m) \tag{4}$$

$$\sum_1^m r_k = O(1)\,, \tag{5}$$

these results being eqs. (29), (30) and (26) of our earlier paper.

LEMMA. *There is a* $\delta > 0$ *such that* $|r_m + r_{m+1}| \leqq 2 - \delta$ *for all sufficiently large* m.

PROOF. From (3) and (4) we obtain

$$m\,|r_m + r_{m+1}| \leqq \sum_{k \leqq m} |r_k + r_{k+1}| + O(\log m)\,.$$

(*) Presentata nella seduta del 13 gennaio 1990.

Suppose, contrary to the conclusion of the lemma, that $|r_m + r_{m+1}| > 2 - \delta$ and let I be the set of indices k such that $|r_k + r_{k+1}| \leqq 2 - \varepsilon$ and $k \leqq m$. In any case we have $|r_k + r_{k+1}| \leqq 2 + O(1/k)$, hence $m(2-\delta) \leqq (2-\varepsilon)|I| + 2(m - |I|) + O(\log m)$. We choose $\varepsilon = \sqrt{\delta}$ and get $|I| \leqq \sqrt{\delta} m + O(\log m)$. This means that $|r_k + r_{k+1}| > 2 - \sqrt{\delta}$ for all but $\sqrt{\delta} m + O(\log m)$ exceptions k, and it follows that if m is large enough there is an interval $K = [k_0, k_1]$, with $m/2 < k_0 < m$, and $|K| > 1/(3\sqrt{\delta})$, such that

$$|r_k + r_{k+1}| > 2 - \sqrt{\delta} \qquad \text{for } k \in K. \tag{6}$$

Since $|r_k| \leqq 1 + O(1/k)$, the last displayed inequality shows that r_k and r_{k+1} have the same sign for $k \in K$. Hence

$$\left| \sum_{k \in K} r_k \right| \geqq (2 - \sqrt{\delta})\,|K| > (2 - \sqrt{\delta})/(3\sqrt{\delta}),$$

which contradicts (5) if δ is small enough.

Let $\limsup |r_m| = A \leqq 1$. Now (3) yields $A \leqq A^2$ hence either $A = 0$ or $A = 1$. If $A = 0$ there is nothing to prove, therefore we may assume that $\limsup |r_m| = 1$. By (3) and (4) we infer

$$m\,|r_m| \leqq \sum_{k=1}^{m-1} |r_k| + O(\log m).$$

Let ε be small and let m be large such that $|r_m| > 1 - \varepsilon$. If we argue as in the proof of the lemma we see that $|r_k| > 1 - \sqrt{\varepsilon}$ for $k \leqq m$, except possibly for a set of cardinality at most $\sqrt{\varepsilon} m + O(\log m)$, and again we see that there is an interval $K = [k_0, k_1]$ such that $m/2 < k_0 < m$, $|K| > 1/(3\sqrt{\varepsilon})$ and $|r_k| \geqq 1 - \sqrt{\varepsilon}$ for $k \in K$. This inequality and the preceding lemma show that if $\varepsilon < \delta^2/4$ then the r_k's must alternate in sign throughout the interval K. It follows that

$$\left| \sum_{2i \in K} r_{2i} \right| \geqq (1 - \sqrt{\varepsilon})(|K|/2 - 1) > (1 - \sqrt{\varepsilon})/(6\sqrt{\varepsilon}) - 1. \tag{7}$$

So far we have not used hypothesis (2) of Theorem 1. However it is clear that $a_{2m} \geqq 0$ and (2) yield the analogue of (5), namely

$$\left| \sum_{k=1}^{m} r_{2k} \right| = O(1),$$

which contradicts (7) if ε is small enough. Thus the hypothesis $A = 1$ is untenable and the proof of Theorem 1 is complete.

To prove Theorem 1 of our earlier paper it remains to verify condition (2). However the statement of condition (2) is nothing else than (24) of our earlier paper applied not to the curve C defined over the finite field $[q]$ with q elements, but rather to the curve C defined now over the field $[q^2]$. Thus Theorem 1 of our earlier paper is a formal consequence of Selberg's Lemma for the curve C over the field $[q]$ and the curve C over the field $[q^2]$.

We remark that a refinement of the above argument can be used to show that the asymptotic formula $r_m + r_{m+1} \sim 1$ holds, hence $a_m + a_{m+1} \sim 2$, as a consequence of (1) and the positivity of a_m; it is an open problem whether these hypotheses suffice to prove that either $a_m \sim 1$ or $a_m \sim 1 + (-1)^{m-1}$.

References

[1] E. Bombieri, *Sull'analogo della formula di Selberg nei corpi di funzioni.* Atti Acc. Lincei Rend. fis., s. 8, vol. 35, fasc. 5, 1963, 252-257.

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