Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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Finite groups with an automorphism of prime order whose fixed points are in the Frattini of a nilpotent subgroup

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **1** (1990), n.2, p. 89–92.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1990_9_1_2_89_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1990.

Rend. Mat. Acc. Lincei
s. 9, v. 1:89-92 (1990)

Teoria dei gruppi. — *Finite groups with an automorphism of prime order whose fixed points are in the Frattini of a nilpotent subgroup.* Nota di ANNA LUISA GILOTTI, presentata (*) dal Socio G. ZAPPA.

ABSTRACT. — In this paper it is proved that a finite group G with an automorphism α of prime order r, such that $C_G(\alpha) = 1$ is contained in a nilpotent subgroup H, with $(|H|, r) = 1$, is nilpotent provided that either $|H|$ is odd or, if $|H|$ is even, then r is not a Fermat prime.

KEY WORDS: Nilpotent subgroup; Automorphism; Simple group; Solvable group.

RIASSUNTO. — *Gruppi finiti dotati di un automorfismo di ordine primo i cui punti fissi sono nel sottogruppo di Frattini di un sottogruppo nilpotente.* In questa nota si prova che un gruppo finito dotato di un automorfismo di ordine primo r, il cui centralizzante è nel sottogruppo di Frattini di un sottogruppo nilpotente H, è nilpotente nell'ipotesi che $(|H|, r) = 1$ ed H sia dispari, oppure se $|H|$ è pari r non sia un primo di Fermat.

INTRODUCTION

In this paper the following result is proved:

THEOREM. *Let G be a finite group and let α be an automorphism of G of prime order r. Suppose that $C_G(\alpha) \leq \Phi(H)$ where H is a nilpotent subgroup of G such that $H^\alpha = H$ and $(|H|, r) = 1$. Then G is nilpotent, provided that either $|H|$ is odd, or if $|H|$ is even, r is not a Fermat prime.*

This theorem generalizes Theorem A [1], where the same result was obtained under the hypothesis $H = P, P$ a Sylow p-subgroup of G $(p \neq r)$ and $C_G(\alpha) = \Phi(P)$.

Although we apply the results of [1] to prove this theorem, the «solvability of G», which in [1] was obtained by a direct argument, is here deduced by using the classification of the finite simple groups.

PRELIMINARY RESULTS

For the convenience of the reader, we begin by recalling some well-known results, which will be used in our proofs, sometimes without specific reference.

All groups considered here are finite.

LEMMA 1.1. *Let A be a group of automorphisms of a finite group G such that $(|A|, |G|) = 1$. If H is a normal A-invariant subgroup of G, $C_{G/H}(A) = C_G(A)H/H$.*

PROOF. See [2, Theorem 6.2.2.].

LEMMA 1.2. *Let α be an automorphism of G of prime order r. If $C_G(\alpha)$ is an r'-group, G is an r'-group.*

(*) Nella seduta del 18 novembre 1989.

PROOF. See [3, Lemma 2.3.i.].

LEMMA 1.3. i) *If N is a normal subgroup of G, $\Phi(N) \leqslant \Phi(G)$.* ii) *If $G/\Phi(G)$ is nilpotent, so is G.*

PROOF. See [2].

LEMMA 1.4. *Let G be a finite group and let α be an automorphism of G of prime order r such that $C_G(\alpha) \leqslant \Phi(P)$, where P is a Sylow p-subgroup of G, with $p \neq r$. Assume further that G has a normal Sylow p-complement K and that either p is odd, or, if $p = 2$, r is not a Fermat prime. Then G is nilpotent.*

PROOF. See [1, Theorem 2.1. and Corollary 2.2.].

RESULTS

LEMMA 2.1. *Let G be a solvable group and let α be an automorphism of G such $C_G(\alpha) \leqslant \Phi(P)$, where P is a Sylow p-subgroup of G. Suppose that α has order r, where r is a prime different from p, and that either p is odd, or, if $p = 2$, r is not a Fermat prime. Then G is nilpotent.*

PROOF. We argue by induction on the order of G. If $O_{p'}(G \neq 1$, then $G/O_{p'}(G)$ satisfies the same hypotheses as G. In fact the hypothesis on (α, G, P) holds also for the factor groups, by Lemma 1.1. and by observing that, if $N \trianglelefteq G$, $\Phi(PN/N) \geqslant \Phi(\mathrm{P})\, N/N$.

Then $G/O_{p'}(G)$ is nilpotent by induction. This implies that G has a normal Sylow p-complement. Then, by Lemma 1.4., G is nilpotent. Thus we may assume that $O_{p'}(G) = 1$ and so $O_p(G) \neq 1$. By inductive hypothesis, $G/O_p(G)$ is nilpotent, so $F(G/O_p(G)) = G/O_p(G)$. It follows that $G = O_{pp'}(G)$ so that $P = O_p(G)$. But then $\Phi(P)$ is normal in G and so, by Lemma 1.3. i) $\Phi(P) \leqslant \Phi(G)$. By Thompson's theorem then $G/\Phi(G)$ is nilpotent and so G is nilpotent.

THEOREM 2.2. *Let G be a solvable group and let α be an automorphism of G of prime order r. Suppose that $C_G(\alpha) \leqslant \Phi(H)$, where H is a nilpotent subgroup of G such that $H^\alpha = H$ and $(|H|, r) = 1$. Then G is nilpotent provided that either $|H|$ is odd or, if $|H|$ is even, r is not a Fermat prime.*

PROOF. By Lemma 1.2. G is an r'-group. As before observe that the hypothesis on (G, H, α) is inherited by the factor groups. Suppose that G has two minimal normal subgroups N_1 and N_2 such that $N_1 \neq N_2$.

By induction G/N_1 and G/N_2 are nilpotent and so $G/N_1 \cap N_2 \lesssim G/N_1 \times G/N_2$ is nilpotent too. Thus we may assume that G has a unique minimal normal subgroup N which is an elementary abelian q-group for some prime q. It follows also that $F(G) = O_q(G) \geqslant N$.

Since $F(G)\, H$ is an α-invariant subgroup of G, if $F(G)\, H < G$, by induction we have $F(G)\, H$ nilpotent. Since $C_G(F(G)) \leqslant F(G)$, this implies $F(G)\, H$ q-group. But then H is a q-group and so H is subnormal in a Sylow q-subgroup Q of G. Also $\Phi(H) \leqslant \Phi(Q)$, by Lemma 1.3.i).

By Lemma 2.1. we get G nilpotent. Therefore we may assume $F(G)H=G$.

Let p be a prime different from q dividing $|H|$. Let P be a Sylow p-subgroup of H; P is obviously a Sylow p-subgroup of G too.

Let us consider $F(G)P=T$. We have T nilpotent and so, if $P\neq 1$, $C_p(F(G))\neq 1$, a contradiction. It follows then $P=1$, that means G a q-group. This concludes the proof.

The hypothesis «G solvable» in Theorem 2.2. can be removed, by using the classification of finite simple groups.

Proposition 2.3. *Let G be a finite group with an automorphism α of prime order r such that $(r,|G|)=1$. Suppose that $C_G(\alpha)$ is a nilpotent subgroup of G. Then G cannot be simple.*

Proof.. Let us assume G simple. It follows $r\neq 2$. The group of outer automorphisms of G, $\mathrm{Out}(G)$, is a 2-group for alternating and sporadic groups (see [4. p. 169]).

So we can assume that G is a Chevalley group, say $G=G(q)$, where $q=p^s$, p a prime ($p=$ characteristic). The hypothesis $(r,|G|)=1$ implies in this case that α is a field automorphism. So $C_G(\alpha)=G(p)$ the corresponding Chevalley group on the base field $GF(p)$.

From this the contradiction.

Proposition 2.4. *Let G be a finite group with an automorphism α of order r, such that $C_G(\alpha)=H$, where H is a nilpotent subgroup of G, such that $(r,|H|)=1$. Then G is solvable.*

Proof. By Lemma 1.2., G is an r'-group. Suppose, by way of contradiction, that G is not solvable and let G be a minimal counterexample. Let N be a proper characteristic subgroup of G. Since N and G/N verify the same hypothesis of G, if $N\neq 1$, we have, by the minimality of G, N and G/N solvable. This implies G solvable, a contradiction.

So we can assume G characteristically simple. Since G is not solvable, G is the direct product of finitely many copies of a simple non-abelian group S: $G=S_1\times \times S_2\times\ldots\times S_n$ where $S_i\cong S$.

We want to show that $S_i^\alpha=S_i$, for each $i=1,2,\ldots,n$.

Suppose, w.l.o.g., $S_1\neq S_1^\alpha$. α permutes the S_i's. Consider the subgroups $S_1, S_1^\alpha,\ldots,S_1^{\alpha^{r-1}}$. They are distinct and permutable elementwise, so consider the subgroup $T=S_1\times \times S_1^\alpha\times\ldots\times S_1^{\alpha^{r-1}}$.

The subgroup $D=\{ss^\alpha\ldots s^{\alpha^{r-1}}|s\in S_1\}\cong S_1\cong S$ is fixed elementwise by α, so that $C_G(\alpha)\geqslant D$. This contradicts the nilpotency of $C_G(\alpha)$. So $S_i^\alpha=S_i$, for each $i=1,2,\ldots,n$.

But then, Proposition 2.3 gives the final contradiction.

Collaboratrice del G.N.S.A.G.A. del C.N.R. Lavoro parzialmente finanziato con fondi M.P.I. 40%.

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