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Fisica matematica. — *On a supplementary conservation law for a hyperbolic model of heat conductor.* Nota di MARIANO TORRISI e ANTONINO VALENTI, presentata (*) dal Socio G. GRIOLI.

ABSTRACT. — In the context of the wave propagation theory in nonlinear hyperbolic systems, we analyse, in the case of a rigid heat conductor, the model proposed by G. Grioli. After introducing the constitutive relations according to the point of view of the extended thermodynamics, we look for the compatibility of the governing equations with a supplementary conservation law. We obtain the functional form of the constitutive quantities and we are able to show that the governing equations may be written in symmetric and conservative form so that the Cauchy problem results well posed.

KEY WORDS: Thermodynamics; Hyperbolic systems; Convexity.

RIASSUNTO. — *Su una legge di conservazione supplementare per un modello iperbolico di un conduttore di calore.* Nell'ambito della teoria della propagazione ondosa per i sistemi iperbolici non lineari si analizza, nel caso di un conduttore rigido, un modello per la propagazione del calore proposto da G. Grioli. Introdotte le relazioni costitutive in accordo con il punto di vista della termodinamica estesa, si ricerca la compatibilità delle equazioni di campo con una legge di conservazione supplementare. Si ottengono le espressioni funzionali delle quantità costitutive e si dimostra che il modello studiato può essere scritto sotto forma simmetrica e conservativa assicurando così la buona posizione del problema di Cauchy.

1. INTRODUCTION

It is well known that the Fourier's law of heat conduction leads to infinite propagation speed for thermal waves. So, several approaches were developed with the aim of extending the classical theory of irreversible thermodynamics in order to account for finiteness of the velocity of propagation of thermal disturbances. A large body of references on this subject can be found in [1-3].

In 1980 G. Grioli [4, 5], within the context of the classical thermodynamics proposed a law of heat propagation which removes the previous paradox and generalizes the Cattaneo-Maxwell law [6, 7].

Recently [8], along the well established lines of the extended irreversible thermodynamics according to which the constitutive relations may, at non equilibrium, depend also on quantities vanishing at equilibrium, the afore-mentioned Author assumes the free energy to be a functional depending on the heat flux as a non equilibrium field variable. In this context He verifies the thermodynamics compatibility of a heat propagation law of the form

$$\psi \dot{\boldsymbol{q}} + m\boldsymbol{q} + T^{-1}\boldsymbol{g} = \boldsymbol{0} \tag{1.1}$$

where $\boldsymbol{q}$ is the heat flux, T is the absolute temperature, $\boldsymbol{g}$ is the temperature gradient, ψ and m are, in general, non-costant coefficients with $m > 0$ and the upper dot represents the material derivative.

(*) Nella seduta del 13 gennaio 1990.

Setting

(1.2) $$\psi = z/TL\,, \qquad m = (1 - z\dot{L}L^{-1})/TL$$

equation (1.1) reduces to

(1.3) $$z\dot{\boldsymbol{q}} + (1 - z\dot{L}L^{-1})\,\boldsymbol{q} + L\boldsymbol{g} = \mathbf{0}$$

which coincides with his own original heat propagation law [4, 5] when z is a positive constant and L is a constitutive (scalar or matrix) function of T and of some appropriate field variables; in the special case of a rigid conductor L depends on T only.

The equation (1.3) contains, as particular case ($z = 0$), the Fourier's law

(1.4) $$\boldsymbol{q} = -L\boldsymbol{g}$$

where upon it follows that L can be interpretate as the heat conductivity.

In this paper, assuming as field equations for a rigid heat conductor the balance of energy and the (1.3) written in conservative form, we search the compatibility with a supplementary conservation law. The latter, when the sign of the source term is definite, plays the role of the entropy principle in the most general formulation proposed by I. Müller [9].

The procedure used here is the same as that given by Lax and Friedrichs [10] for a generic first order quasi-linear system of equations in conservative form and permits to obtain an explicit characterization of the functional form of the constitutive laws. In our opinion this procedure is more convenient than the Lagrange multipliers one [11] because in this theory, the Lagrange multipliers may depend on the external source and it is necessary to make the assumption that the field variables and the initial data are analytic functions on their arguments. Instead, in our procedure we use a new field which depends on the structure of the system and we need only smooth initial data.

Furthermore it is possible to put the system of governing equations in symmetric and conservative form [10, 12]. So, a general theorem on the well-position of the Cauchy problem (locally) holds, ensuring the existence and the uniqueness of the solution when the initial data are assumed smooth on a space like initial surface [13]. Finally the properties of shock waves shown in [14-17] concerning the increase of entropy across the shock wavefront, the existence of a generating function of the shock and the bounded speeds of the shock propagation, hold.

2. Governing equations

In the case of a rigid heat conductor, let $\boldsymbol{x}$ be the orthogonal coordinates of a point of the body with respect to a fixed reference frame $0\xi_i (i = 1, 2, 3)$ and t be the time coordinate. Thus the balance of the specific internal energy e, in absence of external heat sources, is given by

(2.1) $$\partial_t e + \partial_{x_i} q_i = 0$$

where q_i are the components of the heat flux vector, $\partial_t = \partial/\partial_t$ and $\partial_{x_i} = \partial/\partial_{x_i}$.

In the spirit of extended thermodynamics we consider the heat flux $\boldsymbol{q}$ as independent variable, so we may write the constitutive relation for the specific internal energy as

$$e = e(\boldsymbol{q}, T)\,. \tag{2.2}$$

As evolution equation of the non-equilibrium variable $\boldsymbol{q}$ we add the eq. (1.3) which in the rigid case can be written in the following form of balance law

$$\partial_t(\boldsymbol{q}/L) + (1/z)\,\boldsymbol{g} = (-\,1/z)(\boldsymbol{q}/L) \tag{2.3}$$

with z positive relaxation constant and L constitutive scalar function depending on T only.

The eqs. (2.1) and (2.3) may be written as a first order quasi-linear hyperbolic sistem in generalized conservative form

$$\partial_t \boldsymbol{V} + \partial_{x_j} \boldsymbol{F}^j = \boldsymbol{F} \tag{2.4}$$

where the four-dimensional vectors $\boldsymbol{V}$, $\boldsymbol{F}^j$ and $\boldsymbol{F}$ are given by

$$\boldsymbol{V} := \begin{vmatrix} Q_i \\ e \end{vmatrix}; \quad \boldsymbol{F}^j := \begin{vmatrix} (1/z)\,\delta_{ij}\,T \\ q_j \end{vmatrix}; \quad \boldsymbol{F} := \begin{vmatrix} (-\,1/z)\,Q_i \\ 0 \end{vmatrix} \tag{2.5}$$

with

$$\boldsymbol{Q} := \boldsymbol{q}/L \tag{2.6}$$

and δ_{ij} is the Kronecker symbol.

3. On the supplementary conservation law

We require, now, the system (2.4) to be compatible with a supplementary conservation law of the type

$$\partial_t h + \partial_{x_i} h^i = r \tag{3.1}$$

assuming

$$h = h(\boldsymbol{q}, T)\,, \qquad h^i = h^i(\boldsymbol{q}, T)\,. \tag{3.2}$$

The condition (3.1) is equivalent, by making the assumption $r \geq 0$, to the generalized form of the entropy inequality [9].

By way of convenience and in order to get a necessary and sufficient condition of compatibility of (3.1) with the system (2.4), we choose as new field variables

$$\widetilde{\boldsymbol{U}} \equiv (q_i, T)\ (^1). \tag{3.3}$$

The local invertibility of the map $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{U})$ requires that $\boldsymbol{\nabla}_U \boldsymbol{V}$ (1) is a non singular matrix; this implies the assumption

$$D := \det \boldsymbol{\nabla}_U \boldsymbol{V} = e_T + (L'/L)\,\boldsymbol{q}\cdot \boldsymbol{e}_q \neq 0\ (^1), \tag{3.4}$$

By following the usual procedure [10, 12] the necessary and sufficient conditions for the compatibility of the equation (3.1) with the system (2.4) are

$$(\boldsymbol{\nabla}_V h)\cdot d\boldsymbol{F}^i = dh^i\,, \qquad (\boldsymbol{\nabla}_V h)\cdot \boldsymbol{F} = r \tag{3.5}$$

(1) The upper tilde denotes transposition; ∇_ξ denotes gradient with respect to the field ξ; $(\cdot)_\xi = \partial/\partial\xi$ and the prime denotes derivative with respect to the only field variable upon which the function depends.

which, taking (2.5) and (2.6) into account, become

(3.6) $$dh^i = (1/z)\,\Omega_i\,dT + \Gamma\,dq_i$$

(3.7) $$(1/z)(q_i/L)\,\Gamma_i = -r$$

where

(3.8) $$(\Omega_i, \Gamma) := (h_{Q_i}, h_e) = \widetilde{\nabla}_V h\,.$$

From (3.6) we obtain

(3.9) $$h^i_{q_j} = \delta_{ij}\Gamma\,, \qquad h^i_T = (1/z)\,\Omega_i\,.$$

From (3.9.I) follows

(3.10) $$\Gamma = \Gamma(T)\,, \qquad h^i = \Gamma q_i + \hat{h}^i(T)$$

where $\hat{h}^i$ are arbitrary integration functions. Moreover, regarding h^i as components of entropy flux, $\hat{h}^i$ must be taken equal to zero because h^i vanish at equilibrium (*i.e.* $\boldsymbol{q} = \mathbf{0}$, $\boldsymbol{g} = \mathbf{0}$); so taking (3.10) into account in (3.9.II) we obtain

(3.11) $$\Omega_i = z\Gamma'\,q_i\,.$$

After (3.11), (3.7) becomes

(3.12) $$(\Gamma'/L)\,\boldsymbol{q}^2 = -r$$

which implies $\Gamma' \neq 0$ when $r \neq 0$.

By means of (2.7), (3.8) and (3.11) we can write

(3.13) $$dh = (\Gamma e_T - z(L'\Gamma'/L^2)\,\boldsymbol{q}^2)\,dT + (\Gamma e_{\boldsymbol{q}} + z(\Gamma'/L)\,\boldsymbol{q})\cdot d\boldsymbol{q}\,.$$

The integrability conditions arising from (3.13), after some calculations, allows us to obtain

(3.14) $$e_{\boldsymbol{q}} = -(z/L\Gamma')(L'\Gamma'/L + \Gamma'')\,\boldsymbol{q}$$

which integrated gives rise to

(3.15) $$e = e_0(T) - (z/2L\Gamma')(L'\Gamma'/L + \Gamma'')\,\boldsymbol{q}^2$$

where e_0 is the equilibrium specific internal energy.

By introducing (3.15) in (3.13) and integrating we obtain

(3.16) $$h = h_0(T) + (z/2L)(\Gamma' - L'\Gamma/L - \Gamma\Gamma''/\Gamma')\,\boldsymbol{q}^2$$

where

(3.17) $$h_0 = \int_{T_0}^{T} \Gamma e_0'\,dT$$

with T_0 the temperature in the state reference.

Requiring, as is usual at equilibrium, e_0' and h_0' to satisfy the Gibbs relation

(3.18) $$h_0' = T^{-1}\,e_0' > 0\,,$$

from (3.17) it follows

(3.19) $$\Gamma = T^{-1}\,.$$

Then (3.15) and (3.16) can be written as

$$e = e_0 + (z/2L)(2/T - L'/L)\,\boldsymbol{q}^2 \tag{3.20}$$

and

$$h = h_0 + (z/2LT)(1/T - L'/L)\,\boldsymbol{q}^2 . \tag{3.21}$$

Moreover taking the assumption $r \geq 0$ into account introducing (3.19) into (3.12) we get

$$L > 0 . \tag{3.22}$$

It is worth noticing that by assuming *a priori* the internal energy to depend on the temperature only the compatibility of the governing system with a supplementary conservation law requires a special form of L, *i.e.* $L = KT^2$ as was shown in [18], while in the present case apart from (3.22) no restrictions to the functional form of the heat conductivity L arise as it happens for the classical Fourier model.

4. Convexity of $(-h)$

We now look for the conditions for which the convexity of $(-h)$ holds.

The function $(-h)$ is convex if the quadratic form

$$\delta^2 h = (\partial^2 h/\partial \boldsymbol{V}\partial \boldsymbol{V})\,\delta\boldsymbol{V}\cdot\delta\boldsymbol{V} = \delta(\widetilde{\boldsymbol{\nabla}}_V h)\cdot\delta\boldsymbol{V}(<0)(\forall\, \delta\boldsymbol{V}) \tag{4.1}$$

is definite negative.

Taking (3.8), (3.11) and (3.19) into account, from (4.1), after some calculations we get

$$-D^{-1}[z(L'T - 2L)\,T^{-2}\,\boldsymbol{Q}\cdot\delta\boldsymbol{Q} + T^{-1}\delta e]^2 - zLT^{-2}(\delta\boldsymbol{Q})^2 < 0 . \tag{4.2}$$

The form (4.2) is negative definite if and only if

$$D =: e_0' + (z/2)[(L'/L^2)(2/T - L'/L) + (1/L)(2/T - L'/L)']\,\boldsymbol{q}^2 > 0 . \tag{4.3}$$

So, $(-h)$ is convex if the relation (4.3) is satisfied; therefore by means of a suitable Legendre transformation of the field variable $\boldsymbol{V}$[12], the system of eq. (2.4) can be written in symmetric and conservative form.

Consequently the system is hyperbolic and the Cauchy problem results well posed [13] when the initial data are assumed smooth on a space like initial surface. Moreover the properties of shock waves concerning the increasing of the entropy across the shock and the boundedness of the shock speeds, hold [14-17].

Of course if $(L'/L^2)(2/T - L'/L) + (1/L)(2/T - L'/L)' < 0$, then (4.3) implies a bound for $\boldsymbol{q}^2$. Such a kind of situation where the hyperbolicity is guaranted in suitable domains for the field variables naturally arises several non-linear governing models of physical interest [19-22].

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