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Meccanica. — *On a functional equation arising in the kinetic theory of gases.* Nota di
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ABSTRACT. — The problem of finding the summational collision invariants for the Boltzmann equation leads to a functional equation related to the Cauchy equation. The solution of this equation is known under different assumptions on its unknown ψ. Most proofs assume that the equation is pointwise satisfied, while the result needed in kinetic theory concerns the solutions of the equation when the latter is satisfied almost everywhere. The only results of this kind appear to be due to the authors of the present paper. Here the problem is tackled with the aim of giving a simple proof that the most general solution of the problem is not different from the standard one when the equation is satisfied almost everywhere in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$ and ψ is assumed to be measurable and finite a.e.

KEY WORDS: Kinetic theory; Collision invariants; Functional equations.

RIASSUNTO. — *Su un'equazione funzionale che si presenta nella teoria cinetica dei gas.* Il problema di trovare gli invarianti d'urto per l'equazione di Boltzmann porta a un'equazione funzionale legata all'equazione di Cauchy. La soluzione di questa equazione è nota sotto diverse ipotesi sull'incognita ψ. La maggior parte delle dimostrazioni sono basate sull'ipotesi che l'equazione sia soddisfatta puntualmente, mentre il risultato richiesto nella teoria cinetica riguarda le soluzioni dell'equazione quando questa è soddisfatta quasi ovunque. I soli risultati di questo tipo sembrano dovuti agli autori del presente lavoro. Qui si affronta il problema con lo scopo di dare una dimostrazione semplice del fatto che la soluzione più generale non è diversa da quella usuale anche quando si ritiene l'equazione soddisfatta quasi ovunque in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$ e ψ misurabile e finita quasi ovunque.

1. INTRODUCTION

One of the basic ingredients of the kinetic theory of a monatomic rarefied gas [1,2] is the concept of a (summational) collision invariant, *i.e.* a function $\psi(\xi)$ such that

$$\psi(\xi'_*) + \psi(\xi') - \psi(\xi_*) - \psi(\xi) = 0 \tag{1.1}$$

where ξ, ξ_* are vectors in $\mathbb{R}^3$ (with the physical meaning of molecular velocities) and ξ' and ξ'_*, are vectors in $\mathbb{R}^3$ such that

$$\xi'_* + \xi' = \xi_* + \xi\,, \qquad |\xi'_*|^2 + |\xi'|^2 = |\xi_*|^2 + |\xi|^2\,. \tag{1.2}$$

There are several possible representations of ξ' and ξ'_* in terms of ξ_* and ξ; one of the most common ones [1,2], is

$$\xi' = \xi - \boldsymbol{n}(\boldsymbol{n}\cdot \boldsymbol{V})\,, \qquad \xi'_* = \xi_* + \boldsymbol{n}(\boldsymbol{n}\cdot \boldsymbol{V})\,. \tag{1.3}$$

Here $\boldsymbol{V} = \xi - \xi_*$ is the relative velocity and $\boldsymbol{n}$ is a unit vector. Eq. (1.1) must be satisfied almost everywhere in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$.

Eq. (1.1) plays an important role in several problems of kinetic theory; in particular, if f is the distribution function and $\psi = \log f$ then eq. (1.1) must be satisfied by all the possible equilibrium solutions.

(*) Nella seduta del 18 novembre 1989.

The first discussion of eq. (1.1) is due to Boltzmann [3, 4], who assumed ψ to be differentiable twice and arrived at the result that the most general solution of eq. (1.1) is given by

$$\psi(\xi) = A + \boldsymbol{B} \cdot \xi + C|\xi|^2, \tag{1.4}$$

where $A \in \boldsymbol{R}$, $\boldsymbol{B} \in \boldsymbol{R}^3$, $C \in \boldsymbol{R}$ are arbitrary constants. This result seems to be physically obvious and it is remarkable that Boltzmann was not satisfied with physical evidence and felt the necessity of giving the above-mentioned proof.

After Boltzmann, the matter of finding the solutions of eq. (1.1) was investigated by Gronwall [5, 6] (who was first to reduce the problem to Cauchy's functional equation for linear functions), Carleman [7] and Grad [8]. All these authors assumed ψ to be continuous and proved that it must be of the form given in eq. (1.4). Slightly different versions of Carleman's proof are given in refs. [2, 9]. In the latter monograph [9] the authors prove that the solution is of the form (1.4), even if the function ψ is assumed to be measurable rather than continuous. In fact they use a result on the solutions of Cauchy's equation:

$$f(\boldsymbol{u} + \boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v}) \qquad (\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n \text{ or } \mathbb{R}_+) \tag{1.5}$$

valid for measurable functions. It seems, however, that when passing from continuous to (possibly) discontinuous functions, one should insist on the fact that eq. (1.1) is satisfied almost everywhere and not everywhere in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$, as assumed in ref. [9]. It is possible, although this will not be attempted in this paper, to transform the proof in ref. [9] into a proof that the collision invariants are the classical ones under the assumption that eq. (1.1) holds almost everywhere.

In a recent paper [10] the problem of solving eq. (1.1) was tackled with the aim of proving that eq. (1.3) gives the most general solution of eq. (1.1), when the latter is satisfied almost everywhere in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$, under the assumption that ψ is in the Hilbert space H_ω of the square integrable functions with respect to a Maxwellian weight $\omega(|\xi|) = (\beta/\pi)^{3/2} \exp(-\beta|\xi|^2)$, $\beta > 0$. The first step was to show that the linear manifold of the solutions possessed a polynomial basis. After that it was enough to look for smooth solutions. The existence of these can be made very simple if we look for C^2 solutions.

A completely different proof of the same result (under the assumption that $\psi \in L^1_{\text{loc}}$) is contained in a paper by Arkeryd [11]. Since this is not widely known and Arkeryd's arguments, when combined with the proof for C^2 functions of ref. [10], allow a very simple proof of the fact that (1.4) is the most general solution when $\psi \in L^1_{\text{loc}}$ and eq. (1.1) is satisfied almost everywhere, we shall discuss this proof in some detail.

Following Arkeryd [11] we shall use another representation of ξ'_* and ξ'; we let

$$\begin{aligned} \xi' &= \xi + \boldsymbol{u} \\ \xi'_* &= \xi + \boldsymbol{v} \end{aligned} \tag{1.6}$$

so that because of eqs. (1.2)

$$\xi_* = \xi + \boldsymbol{u} + \boldsymbol{v} \tag{1.7}$$

and

$$u \cdot v = 0 . \tag{1.8}$$

Eq. (1.1) then becomes:

$$\psi(\xi + u + v) + \psi(\xi) = \psi(\xi + u) + \psi(\xi + v) \tag{1.9}$$

which must be satisfied *a.e.* on the manifold $u \cdot v = 0$ of $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$.

The connection of eq. (1.9) with Cauchy's equation is now easily seen if we (arbitrarily) fix ξ and let

$$f(u) = \psi(\xi + u) - \psi(\xi) . \tag{1.10}$$

In fact. eq. (1.9) immediately becomes:

$$f(u + v) = f(u) + f(v) \qquad (u \cdot v = 0) \tag{1.11}$$

which is a restriction of eq. (1.5) with $n = 3$ to a subset of $\mathbb{R}^3 \times \mathbb{R}^3$.

We remark that the problem makes sense in any $\mathbb{R}^n$ for any $n \geq 2$, although we stick to the case $n = 3$ for convenience.

2. Smooth solutions can be constructed in terms of (possibly) non-smooth solutions

In Arkeryd's paper [11] the trick to solve eq. (1.11) when it is satisfied almost everywhere and f is in L^1_{loc}, was to introduce an auxiliary function

$$g(u) = \int_0^1 f(tu)\, dt \tag{2.1}$$

which turns out to be C^0. It is clear that eq. (1.11) implies

$$g(u + v) = g(u) + g(v) \qquad (u \cdot v = 0) \tag{2.2}$$

so that we construct C^0 solutions in terms of solutions that are only L^1_{loc}. In this section we extend his procedure in order to show that we can construct n times differentiable solutions in terms of $n - 1$ times differentiable solutions ($n \geq 1$) We first start with a proof of Arkeryd's result:

LEMMA 1. If $f \in L^1_{\text{loc}}$ satisfies eq. (1.11) then eq. (2.1) makes sense and $g \in C^0$.

PROOF. We introduce an orthonormal basis e_i ($i = 1, 2, 3$) in R^3 and write $u = \sum_i u_i e_i$. Then, because of eq. (1.11):

$$f(u) = \sum_i f(u_i e_i) \tag{2.3}$$

if f is continuous. A similar situation prevails if f is measurable (in particular is in L^1_{loc}) and eq. (1.11) holds *a.e.* $(u, v) \in \mathrm{M} = \{u, v \in R^3;\ u \cdot v = 0\}$. Namely, for *a.e.* choice of e_1 then eq. (1.11) implies

a) for *a.e.* choice of e_2, e_3 orthogonal to e_1 and of $(u_2, u_3) \in \mathbb{R}$, that

$$f(u_2 e_2 + u_3 e_3) = f(u_2 e_2) + f(u_3, e_3) ; \tag{2.4}$$

b) for *a.e.* choice of $\boldsymbol{q} \in \mathbb{R}^3$ orthogonal to $\boldsymbol{e}_1$ and of $u_1 \in \mathbb{R}$, that

$$f(u_1 \boldsymbol{e}_1 + \boldsymbol{q}) = f(u_1 \boldsymbol{e}_1) + f(\boldsymbol{q}). \tag{2.5}$$

Hence we can pick an orthonormal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ so that for *a.e.* $\boldsymbol{u} = \sum u_j \boldsymbol{e}_j \in \mathbb{R}^3$, eq. (2.3) holds.

Then the following integral exists:

$$\int_0^{u_1} dv_1 \int_0^{u_2} dv_2 \int_0^{u_3} dv_3\, f(\boldsymbol{v}) = \int_0^{u_1} dv_1 \int_0^{u_2} dv_2 \int_0^{u_3} dv_3 \sum_i f(v_i \boldsymbol{e}_i) = \tag{2.6}$$

$$= u_1 u_2 u_3 \sum_i \frac{1}{u_i} \int_0^{u_i} dv_i\, f(v_i \boldsymbol{e}_i) = u_1 u_2 u_3 \sum_i \int_0^1 dt f(tu_i \boldsymbol{e}_i) = u_1 u_2 u_3 \int_0^1 \mathrm{d}t\, \mathrm{f}(t\boldsymbol{u}) = u_1 u_2 u_3\, g(\boldsymbol{u}).$$

This relation shows that, since the first integral is a continuous function of $u_1, u_2, u_3, g(\boldsymbol{u})$ is continuous in $\mathbb{R}^3 - \{0\}$. Then eq. (2.2) holds for $\boldsymbol{u}, \boldsymbol{v} \neq 0$. If we (arbitrarily) fix $\boldsymbol{u}$ and let $\boldsymbol{v}$ go to zero in a direction orthogonal to $\boldsymbol{u}$, eq. (2.2) shows that

$$\lim_{v \to 0} g(\boldsymbol{v}) = 0 \tag{2.7}$$

when $\boldsymbol{v}$ is restricted to a plane, in particular one of the coordinate planes. From here, using (2.3), it follows that (2.5) holds, when $\boldsymbol{v}$ tends to zero in $\mathbb{R}^3$. Thus g is continuous in $\mathbb{R}^3$ and eq. (2.2) holds with no further restriction.

We can now prove that we can construct continuous and n times differentiable solutions in terms of continuous $n-1$ times differentiable solutions ($n \geqslant 1$).

Lemma 2. If f is continuous and $n-1$ times differentiable ($n \geqslant 1$) and satisfies eq. (1.11) then g is n times differentiable and:

$$\boldsymbol{u} \cdot \frac{\partial g}{\partial \boldsymbol{u}} + g = f. \tag{2.8}$$

Proof. First it is clear from eq. (2.2) that g is $n-1$ times differentiable. In order to prove that actually g is n times differentiable and eq. (2.8) holds, we use eq. (2.6) to relate the partial derivatives of g to f. In fact for $\boldsymbol{u} \neq 0$ eq. (2.4) gives

$$g(\boldsymbol{u}) = \sum_i \frac{1}{u_i} \int_0^{u_i} dv_i\, f(v_i \boldsymbol{e}_i) \tag{2.9}$$

and hence

$$\partial g/\partial u_i = u_i^{-1} f(u_i \boldsymbol{e}_i) - u_i^{-2} \int_0^{u_i} dv_i\, f(v_i \boldsymbol{e}_i) = \tag{2.10}$$

$$= u_i^{-1} f(u_i \boldsymbol{e}_i) - u_i^{-1} \int_0^1 dt f(tu_i \boldsymbol{e}_i) = (f(u_i \boldsymbol{e}_i) - g(u_i \boldsymbol{e}_i))\, u_i^{-1}.$$

This relation shows that, since f and g are $n-1$ times differentiable, g is n times differentiable in $\mathbb{R}^3 - \{0\}$. Further since g satisfies eq. (2.2), we have $g(0) = 0$ and, if we fix $\boldsymbol{u} \neq 0$ orthogonal to $\boldsymbol{e}$

$$\lim_{h \to 0} g(h\boldsymbol{e}_i)\, h^{-1} = \lim_{h \to 0} (g(\boldsymbol{u} + h\boldsymbol{e}_i) - g(\boldsymbol{u}))\, h^{-1}. \tag{2.11}$$

The first limit (if it exists) is the partial derivative $\partial g/\partial u_i$ at the origin, the second is the same derivative at $\boldsymbol{u} \neq 0$. Since the second limit exists, the first exists as well. Hence g is differentiable at the origin. Eq. (2.10) (together with eqs. (1.11) and (2.2)) prove that eq. (2.8) holds for $\boldsymbol{u} \neq 0$. Since f, g and $\boldsymbol{u}\cdot\partial g/\partial \boldsymbol{u}$ are zero at the origin, eq. (2.8) holds for $\boldsymbol{u}=0$ as well.

As we shall see in sect. 4 the results of this section can be used to obtain the solutions in L^1_{loc} from those which are n times differentiable.

3. From the Cauchy functional equation to the functional equation for the collision invariants.

In sect. 1 we saw that one can deduce eq. (1.11) from eq. (1.1); having found results on eq. (1.11) we could now transfer them on eq. (1.1). It is worthwhile, however, to proceed to a deeper analysis of the connection between the two equations, which leads to a better understanding of both.

To this end we assume that we are given eq. (1.11); can we discover its relation with eq. (1.1)? The answer is yes as will be shown by

Lemma 3. Let f be a measurable solution of eq. (1.11). Then $\psi = f$ is a solution of eq. (1.1). This holds even if the equations are satisfied *a.e.* in $\mathbb{M}$ and in $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2$, respectively.

Proof. Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{t}$ be three vectors, with $\boldsymbol{u}\cdot\boldsymbol{v}=0$. Let us decompose $\boldsymbol{t}$ as

$$\boldsymbol{t} = \boldsymbol{t}_u + \boldsymbol{t}_v + \boldsymbol{t}_0, \tag{3.1}$$

where $\boldsymbol{t}_u$ and $\boldsymbol{t}_v$ are directed as $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively, while $\boldsymbol{t}_0$ is orthogonal to both. We can restrict $\boldsymbol{u}$ e $\boldsymbol{v}$ in such a way that eq. (1.11) holds for $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{u}+\boldsymbol{v})$ and also when $(\boldsymbol{u}+\boldsymbol{t}_u, \boldsymbol{t}_v+\boldsymbol{t}_0, \boldsymbol{u}+\boldsymbol{t}_u+\boldsymbol{t}_v+\boldsymbol{t}_0)$, $(\boldsymbol{t}_v,\boldsymbol{t}_0,\boldsymbol{t}_v+\boldsymbol{t}_0)$, $(\boldsymbol{v}+\boldsymbol{t}_v, \boldsymbol{t}_u+\boldsymbol{t}_0, \boldsymbol{v}+\boldsymbol{t}_v+\boldsymbol{t}_u+\boldsymbol{t}_0)$, $(\boldsymbol{t}_u,\boldsymbol{t}_0,\boldsymbol{t}_u+\boldsymbol{t}_0)$, $(\boldsymbol{u}+\boldsymbol{t}_u, \boldsymbol{v}+\boldsymbol{t}_v, \boldsymbol{u}+\boldsymbol{t}_u+\boldsymbol{v}+\boldsymbol{t}_v)$, $(\boldsymbol{u}+\boldsymbol{t}_u+\boldsymbol{v}+\boldsymbol{t}_v, \boldsymbol{t}_0, \boldsymbol{u}+\boldsymbol{t}_u+\boldsymbol{v}+\boldsymbol{t}_v+\boldsymbol{t}_0)$, $(\boldsymbol{t}_u,\boldsymbol{t}_v,\boldsymbol{t}_u+\boldsymbol{t}_v)$, $(\boldsymbol{t}_u+\boldsymbol{t}_v,\boldsymbol{t}_0,\boldsymbol{t}_u+\boldsymbol{t}_v+\boldsymbol{t}_0)$ replace $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{u}+\boldsymbol{v})$. Obviously we still have a set of full measure in $\mathbb{M}\times\mathbb{R}^3$. Then

$$f(\boldsymbol{u}+\boldsymbol{t}) = f(\boldsymbol{u}+\boldsymbol{t}_u) + f(\boldsymbol{t}_v) + f(\boldsymbol{t}_0), \tag{3.2}$$

$$f(\boldsymbol{v}+\boldsymbol{t}) = f(\boldsymbol{v}+\boldsymbol{t}_v) + f(\boldsymbol{t}_u) + f(\boldsymbol{t}_0), \tag{3.3}$$

$$\begin{aligned} f(\boldsymbol{u}+\boldsymbol{t}) + f(\boldsymbol{v}+\boldsymbol{t}) &= f(\boldsymbol{u}+\boldsymbol{t}_u) + f(\boldsymbol{v}+\boldsymbol{t}_v) + f(\boldsymbol{t}_0) + f(\boldsymbol{t}_u) + f(\boldsymbol{t}_v) + f(\boldsymbol{t}_0) = \\ &= f(\boldsymbol{u}+\boldsymbol{t}_u+\boldsymbol{v}+\boldsymbol{t}_v+\boldsymbol{t}_0) + f(\boldsymbol{t}_u+\boldsymbol{t}_v+\boldsymbol{t}_0) = f(\boldsymbol{t}+\boldsymbol{u}+\boldsymbol{v}) + f(\boldsymbol{t}) \qquad (\boldsymbol{u}\cdot\boldsymbol{v}=0), \end{aligned} \tag{3.4}$$

i.e. $\psi = f$ satisfies eq. (1.1), in the form (1.9) (with $\boldsymbol{t}$ in place of ξ).

The previous lemma leads to a deeper property of eq. (1.11), in the form of

Lemma 4. If f is a solution of eq. (1.11) in *a.e.* sense, then if $\boldsymbol{u}$ and $\boldsymbol{v}$ are generic vectors (with $\boldsymbol{u}\cdot\boldsymbol{v}\neq 0$, in general), then $f(\boldsymbol{u}) + f(\boldsymbol{v})$ is a function of $\boldsymbol{u}+\boldsymbol{v}$ and $|\boldsymbol{u}|^2+|\boldsymbol{v}|^2$ in *a.e.* sense.

Proof. According to the previous lemma,

$$f(\boldsymbol{u}+\boldsymbol{t}) + f(\boldsymbol{v}+\boldsymbol{t}) = f(\boldsymbol{t}+\boldsymbol{u}+\boldsymbol{v}) + f(\boldsymbol{t}) \qquad (\boldsymbol{u}\cdot\boldsymbol{v}=0) \tag{3.4}$$

in a set of full measure. If we let

$$w = t + u + v \tag{3.5}$$

then

$$f(t) + f(w) = f(t+u) + f(w-u). \tag{3.6}$$

Here t and w are arbitrary vectors and u is chosen in such a way as to satisfy the constraint

$$|t+u|^2 + |w-u|^2 = |t|^2 + |w|^2. \tag{3.7}$$

However $t' = t + u$, $w' = w - u$ with u satisfying eq. (3.7) is the most general transformation leaving both $t + w$ and $|t|^2 + |w|^2$ invariant. Thus eq. (3.6) gives that a function F must exist such that for *a.e.* value of $t + w$, $|t|^2 + |w|^2$

$$f(t) + f(w) = F(t + w, |t|^2 + |w|^2) \tag{3.8}$$

for *a.e.* value of the remaining variables in (3.7).

If we let $t = t_0$, where t_0 is any point for which eq. (3.6) holds for almost any w, we obtain the result.

The results of the previous section indicate that there is a precise relation between smooth solutions and L^1_{loc} solutions. This leads us to expect that the latter can be recovered from the former. To see how this works we investigate C^2 solutions. Because of the connection between eqs. (1.1) and (1.11) that we have just established, we can proceed to a study of eq. (1.1).

LEMMA 5. If $f \in C^2$, then the most general solution of eq. (1.11) is given by

$$f(u) = B \cdot u + C|u|^2. \tag{3.9}$$

We reproduce the proof recently given by one of the authors [10] for eq. (1.1), similar to Boltzmann's argument [3, 4]. We start from the previous lemma, according to which we must have

$$f(u) + f(v) = F(x, y) \tag{3.10}$$

where

$$x = u + v;\ y = 2^{-1}(|u|^2 + |v|^2). \tag{3.11}$$

If we differentiate eq. (3.10) with respect to u and subtract from the result the analogous derivative with respect to v, we obtain:

$$\partial f/\partial u - \partial f/\partial v = (\partial F/\partial y)(u - v), \tag{3.12}$$

where the arguments are made clear by the variables with respect to which a function is differentiated. Eq. (3.12) implies:

$$(\partial f/\partial u_i - \partial f/\partial v_i)(u_k - v_k) = (\partial f/\partial u_k - \partial f/\partial v_k)(u_i - v_i) \quad (i, k = 1, 2, 3). \tag{3.13}$$

If we now differentiate with respect to u_r we obtain

$$(3.14)\qquad (\partial f/\partial u_i - \partial f/\partial v_i)\,\delta_{kr} + (\partial^2 f/\partial u_i\,\partial u_r)(u_k - v_k) = \\ = (\partial f/\partial u_k - \partial f/\partial v_k)\,\delta_{ir} + (\partial^2 f/\partial u_k\,\partial u_r)(u_i - v_i)$$

where δ_{kr} denotes the Kronecker delta. A further differentiation with respect to v_j gives:

$$(3.15)\qquad \partial^2 f/\partial u_i\,\partial u_r\,\delta_{kj} + \partial^2 f/\partial v_i\,\partial v_j\,\delta_{kr} = \partial^2 f/\partial u_k\,\partial u_r\,\delta_{ij} + \partial^2 f/\partial v_k\,\partial v_j\,\delta_{ir}\,.$$

If we let i, k, r take three different values (say $i=1$, $k=2$, $r=3$) and $j=k$, we obtain:

$$(3.16)\qquad \partial^2 f/\partial u_i\,\partial u_r = 0 \qquad (i, r = 1,2,3;\ i \neq r)\,.$$

If we now take $i=r$, $k=j$, $i \neq k$ in eq. (3.15), we obtain:

$$(3.17)\qquad \partial^2 f/\partial u_i^2 = \partial^2 f/\partial v_k^2 \qquad (i \neq k)\,.$$

Since the right hand side cannot depend on $\boldsymbol{u}$ we conclude that both sides are constant; this constant does not depend on the index, because we can change the values of i and k, while keeping $i \neq k$. From eqs. (3.16) and (3.17) we thus conclude that:

$$(3.18)\qquad \partial^2 f/\partial u_i\,\partial u_r = 2C\delta_{ir} \qquad (i, r = 1,2,3;\ C = \text{const})\,.$$

Eq. (3.18) immediately delivers eq. (3.9).

We can now prove the following

THEOREM 6. If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is in L^1_{loc} and satisfies (1.11) for $(\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{M}$, then for some $\boldsymbol{B} \in \mathbb{R}^3$, $C \in \boldsymbol{R}$, eq. (3.9) holds.

In fact we remark that if we apply the transformation

$$(3.19)\qquad g(\boldsymbol{u}) = \int_0^1 f(t\boldsymbol{u})\,dt$$

we obtain a solution g in C^0; by repeating the transformation twice we arrive at a solution g twice differentiable. This according to the previous lemma is given by

$$(3.20)\qquad g(u) = \boldsymbol{B}\cdot\boldsymbol{u} + C|\boldsymbol{u}|^2\,.$$

We have now to invert three times the transformation given by eq. (3.19); however the set of functions of the form (3.20) is invariant with respect to the transformation (3.19), as well as to its inverse, given by eq. (2.8). Hence when we apply the latter equation three times, in order to recover the original L^1_{loc} solutions from thrice differentiable solutions, we find that the former are still given by eq. (3.20).

4. On measurable summational invariants

By the proof of Lemma 2

$$(4.1)\qquad g(\boldsymbol{u}) = \int_0^1 f(t\boldsymbol{u})\,dt$$

is a continuous solution of eq. (1.11), if (1.11) holds for *a.e.* $(\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{M}$, and $f \in L^1_{\text{loc}}$. This together with eqs. (2.9) and (2.10) gives all solutions of (1.11) in L^1_{loc}, once the continuous ones are known. In this way the study of solutions in L^1_{loc} was reduced to the continuous ones in ref. [11]. In the previous two sections we showed how to reduce them to the study of twice differentiable solution. Alternatively, it is possible to prove which the continuous solutions are, in such a way that the case of L^1_{loc} solutions follows by the continuous proof «with *a.e.* added at suitable places».

In fact, the proof we shall now present can be directly used under the weaker assumption that f is measurable, finite *a.e.* and that eq. (1.11) holds for *a.e.* $(\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{M}$. Following Carleman [7] we split f into an even part $k(\boldsymbol{u}) = f(\boldsymbol{u}) + f(-\boldsymbol{u})$, and an odd part $h(\boldsymbol{u}) = f(\boldsymbol{u}) - f(-\boldsymbol{u})$, which separately satisfy eq. (1.11). Carleman's study of k is simple and also hold in the measurable case «with *a.e.* added at suitable places». As for h, his construction uses in an essential way a set of measure zero, not easy adaptable to the measurable case. Here we will use a different strategy, which seems to be both new and simple. We shall prove the following

THEOREM 7. If f: $\mathrm{R}^3 \to \mathrm{R}$ is continuous and satisfies (1.11) for $(\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{M}$, then for some $\boldsymbol{B} \in \mathrm{R}^3$, $C \in \boldsymbol{R}$, it holds that

$$f(u) = \boldsymbol{B} \cdot \boldsymbol{u} + C|\boldsymbol{u}|^2 . \tag{4.2}$$

If f is measurable, finite *a.e.*, and satisfies (1.11) for *a.e.* $(\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{M}$, then (4.2) holds for *a.e.* $\boldsymbol{u} \in \mathrm{R}^3$.

The proof in the continuous case uses Cauchy's result that any continuous function χ satisfying

$$\chi(x) + \chi(y) = \chi(x+y), \qquad x, y \in \mathrm{R} \ (\text{or } \mathrm{R}_+) \tag{4.3}$$

is of the form $\chi(x) = \beta x$ for some $\beta \in \mathrm{R}$.

A generalization of this result can be used to prove the proposition in the measurable case:

LEMMA 8. If χ is a measurable function from R (or R_+), finite *a.e.*, satisfying (4.3) for *a.e.* $(x, y) \in \mathrm{R}^2$ (or R^2_+), then there is $\beta \in \mathrm{R}$ such that $\chi(x) = \beta x$ for *a.e.* $x \in \mathrm{R}$ (or R_+).

PROOF: The idea is to show that $\chi \in L^\infty_{loc}$, and then make a study of $\int_0^1 \chi(xt)\, dt$ as in sect. 2. Let the domain of χ be R.

Given an interval $\mathrm{I} = (-a/2, a/2)$, by Lusin's theorem there is a continuous function F on R, such that $\chi(x) = F(x)$ for all $x \in \mathrm{I}$ outside of a measurable set of measure less than $a/3$. For some $\delta > 0$, $|F(x+h) - F(x)| < 1$ if $|h| < \delta$, $x \in \mathrm{I}$. Take $\delta < a/3$ and notice that for each h with $|h| < \delta$, $\chi(x+h) = F(x+h)$ for all $x \in \mathrm{I}$ outside of a measurable set of measure less than $a/3 + \delta < 2a/3$.

Thus, given h with $|h| < \delta$, there is a subset $\Omega_h \subset \mathrm{I}$ of a measure larger than $a/3 - \delta > 0$, with $|\chi(x) - \chi(x+h)| < 1$ for $x \in \Omega_h$. But for *a.a.* $(x, h) \in \mathrm{I}x(-\delta, \delta)$:

$$\chi(x+h) - \chi(x) = \chi(h) . \tag{4.4}$$

In particular for *a.a.* $h \in (-\delta, \delta)$ there is an $x_0 \in \Omega_h$ such that

$$1 > |\chi|(x_0 + h) - \chi(x_0)| = |\chi(h)| = |\chi(x+h) - \chi(x)| \qquad \text{for } a.e.\ x \in \mathrm{I}. \tag{4.5}$$

Hence by Fubini's theorem it holds for *a.e.* $x \in \mathrm{I}$ that

$$|\chi(x+h) - \chi(x)| < 1 \qquad \text{for } a.a.\ h \text{ with } |h| < \delta. \tag{4.6}$$

It follows that $\chi \in L^\infty(\mathrm{I})$ and, since I is arbitrary, that $\chi \in L^1_{\mathrm{loc}}$. Thus for $x \neq 0$

$$g(x) = \int_0^1 \chi(tx)\, dt = \int_0^x \chi(s)\, ds/x \tag{4.7}$$

is well defined and continuous. With $g(0) = 0$ it satisfies

$$g(x) + g(y) = g(x+y) \qquad \text{for } (x, y) \in \mathbb{R}^2. \tag{4.8}$$

Similarly to sect. 2 it follows that $g(x) = \beta x$ and that $\chi(x) = 2\beta x$ *a.e.*

PROOF OF THEOREM 7. For the even continuous k of (1.11) Carleman [7] noted that

$$k(\boldsymbol{u}) + k(\boldsymbol{v}) = f(\boldsymbol{u} \pm \boldsymbol{v}) + f(-(\boldsymbol{u} \pm \boldsymbol{v})), \qquad (\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{M}. \tag{4.9}$$

In particular for $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \mathbb{R}^3$ with $|\boldsymbol{p}_1| = |\boldsymbol{p}_2| = r$, and $\boldsymbol{u} = (\boldsymbol{p}_1 + \boldsymbol{p}_2)/2$, $\boldsymbol{v} = (\boldsymbol{p}_1 - \boldsymbol{p}_2)/2$, this gives:

$$k(\boldsymbol{p}_1) = k(\boldsymbol{u} + \boldsymbol{v}) = k(\boldsymbol{u} - \boldsymbol{v}) = k(\boldsymbol{p}_2). \tag{4.10}$$

So there is a function ϕ with $k(\boldsymbol{p}) = \phi(r^2)$. Finally, we obtain

$$\phi(|\boldsymbol{p}_1|^2) + \phi(|\boldsymbol{p}_2|^2) = \phi(|\boldsymbol{p}_1|^2 + |\boldsymbol{p}_2|^2) \tag{4.11}$$

and by Cauchy's result we get

$$k(\boldsymbol{u}) = \Phi(|\boldsymbol{u}|^2) = 2C|\boldsymbol{u}|^2, \tag{4.12}$$

where we replaced β by $2C$.

In the measurable case, starting from (1.11) for k and *a.e.* $(\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{M}$, we can argue in the same way and by Lemma 8 conclude that (4.12) holds for *a.e.* $\boldsymbol{u} \in \mathbb{R}^3$.

For the odd function h, in the continuous case we let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ be an arbitrary orthonormal basis in $\mathbb{R}^3$ and notice that (2.3) holds for h and this basis. For $(\boldsymbol{u}, \boldsymbol{v})$ in M set $\boldsymbol{u} = \sum u_j \boldsymbol{e}_j$ and $\boldsymbol{v} = \sum v_j \boldsymbol{e}_j$. By (1.11)

$$\begin{aligned} \sum (h(u_j \boldsymbol{e}_j) + h(v_j \boldsymbol{e}_j)) &= h(\textstyle\sum u_j \boldsymbol{e}_j) + h(\sum v_j \boldsymbol{e}_j) = h(\boldsymbol{u}) + h(\boldsymbol{v}) = h(\boldsymbol{u} + \boldsymbol{v}) = \\ &= h(\textstyle\sum ((u_j + v_j)\, \boldsymbol{e}_j)) = \sum h((u_j + v_j)\, \boldsymbol{e}_j). \end{aligned} \tag{4.13}$$

And so:

$$h(u_1 \boldsymbol{e}_1) + h(v_1 \boldsymbol{e}_1) - h((u_1 + v_1)\, \boldsymbol{e}_1) = -\sum_2^3 (h(u_i \boldsymbol{e}_i) + h(v_i \boldsymbol{e}_i) - h((u_i + v_i)\, \boldsymbol{e}_i)). \tag{4.14}$$

Since h is odd this gives

$$h(u_i \boldsymbol{e}_i) + h(v_i \boldsymbol{e}_i) - h((u_i + v_i)\, \boldsymbol{e}_i) = 0 \qquad (i = 1). \tag{4.15}$$

An analogous result holds for $i=2, 3$. So, by Cauchy's result, for some $B_i \in \mathbb{R}$

$$h(u_i e_i) = 2B_i u_i \tag{4.16}$$

and with $\boldsymbol{B} = \sum B_i \boldsymbol{e}_i$:

$$h(\boldsymbol{u}) = 2\boldsymbol{B} \cdot \boldsymbol{u}. \tag{4.17}$$

By the discussion of eq. (2.3), in the measurable case there is an orthonormal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ such that eq. (2.3) holds for h and *a.e.* $\boldsymbol{u} \in \mathbb{R}^3$. Using this basis, the above discussion holds for *a.e.* $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{M}$, in particular eq. (4.17) holds for almost everywhere $\boldsymbol{u} \in \mathbb{R}^3$. Finally eq. (4.2) follows (for *a.e.* $\boldsymbol{u} \in \mathbb{R}^3$) by adding eqs. (4.12) and (4.17).

5. Concluding remarks

The result given in the previous section is somehow the most general that one can hope for the solutions of eqs. (1.11) and (1.1). An immediate consequence is that if the integrand in the Boltzmann collision operator is zero *a.e.* and the distribution function M is positive and finite *a.e.*, then M is a Maxwellian. It follows from the discussion in ref. [11] that any such density, which is just known to be positive on some set of positive measure, has to be positive *a.e.* Thus we have the following

Corollary. If $M(\boldsymbol{v}): \mathbb{R}^3 \to [0, \infty)$ is measurable, finite, positive on a set of positive measure, and satisfies

$$M(\xi + \boldsymbol{u} + \boldsymbol{v}) M(\xi) = M(\xi + \boldsymbol{u}) M(\xi + \boldsymbol{v}) \quad \text{for } a.e. \ (\xi, \boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^9 \text{ with } (\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{M} \tag{5.1}$$

then there are $a, c \in \mathbb{R}$ and $\boldsymbol{b} \in \mathbb{R}^3$ such that

$$M(\boldsymbol{v}) = a \exp(-c|\boldsymbol{v} - \boldsymbol{b}|^2). \tag{5.2}$$

References

[1] C. Cercignani, *Mathematical Methods in Kinetic Theory.* Plenum Press, New York 1969.

[2] C. Cercignani, *The Boltzmann Equation and Its Applications.* Springer Verlag, New York 1988.

[3] L. Boltzmann, *Über das Wärmegleichgewicht von Gasen, auf welche äussere Kräfte wirken.* Sitzungsberichte der Akademie der Wissenschaften Wien, 72, 1875, 427-457.

[4] L. Boltzmann, *Über die Aufstellung und Integration der Gleichungen, welche die Molekular bewegungen in Gasen bestimmen.* Sitzungsberichte der Akademie der Wissenschaften Wien, 74, 1876, 503-552.

[5] T. H. Gronwall, *A functional equation in the kinetic theory of gases.* Annals of Mathematics, (2) 17, 1915, 1-4.

[6] T. H. Gronwall, *Sur une équation fonctionelle dans la théorie cinétique des gaz.* Comptes Rendus de l'Académie des Sciences, 162, Paris, 1916, 415-418.

[7] T. Carleman, *Problèmes Mathématiques dans la Théorie Cinétique des Gaz.* Almqvist & Wiksell, Uppsala 1957.

[8] H. Grad, *On the kinetic theory of rarified gases.* Communications on Pure and Applied Mathematics, 2, 1949, 331-407.

[9] C. TRUESDELL - R. G. MUNCASTER, *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas.* Academic Press, New York 1980.

[10] C. CERCIGNANI, *Are there more than five linearly independent collision invariant for the Boltzmann equations?.* J. Statistical Phys., to appear, 1990.

[11] L. ARKERYD, *On the Boltzmann equation. Part II: The full initial value problem.* Arch. Rat. Mech. Anal., 45, 1972, 17-34.

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