Multivalued nonpositone problems

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Abstract. — In this note, the existence of non-negative solutions for some multivalued non-positone elliptic problems is studied.

Key words: Elliptic multivalued problem; Discontinuous nonlinearities; Sub-linear and superlinear.

Riassunto. — Problemi di tipo «non-positone» a multivalori. In questa nota si studia la esistenza di soluzioni non negative di certi problemi a multivalori ellittici non lineari.

0. INTRODUCTION

In this paper, we will consider the boundary value problem,

\[ -\Delta u = f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) and \( f: [0, +\infty) \rightarrow \mathbb{R} \) is a \( C^1 \)-function with \( f(0) < 0 \) (non-positone).

Recently, Brown et al. [4] have proved a result of non-existence of non-negative radial solutions of (0.1), when \( \Omega \) is a ball and \( f \) is a superlinear and increasing function. In concrete, it is proved there that, if \( f = \lambda g \) with \( \lambda \in \mathbb{R} \), then there exists \( \lambda_0 > 0 \) such that (0.1) has no such solutions for all \( \lambda \geq \lambda_0 \). For existence of at least one positive solution for \( \lambda \) sufficiently small, see [5].

Motivated by this result, we will study here the existence of non-negative solutions of the multivalued problem

\[ -\Delta u(x) \in \bar{f}(u(x)) \quad \text{a.e. } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \quad u \geq 0 \quad \text{in } \Omega \]

where \( \bar{f} \) is the multivalued function defined by

\[ \bar{f}(u) = \begin{cases} [f(0), 0], & \text{if } t = 0; \\ f(t), & \text{if } t > 0. \end{cases} \]

In contrast with [4], we will prove the existence of i) one non-zero \( C^1 \)-solution of (0.2) if \( f \) is superlinear (with no further restrictions); ii) two non-zero and distinct solutions of (0.2) if \( f \) is asymptotically linear (not at resonance) verifying some additional condition.

There have been some works on elliptic problems with discontinuous nonlinearities where a suitable direct variational approach is used ([1], [6] and [10]). However, here we find more convenient (at least, in the superlinear case) to work on the approximating problems

\[ -\Delta u = f_n(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega \]

where \( f_n \) is a sequence of functions which «converges» in some sense to \( f \) and \( f_n(0) = 0 \). A convenient choice of \( f_n \) permits us to prove the existence of solutions of (0.3), which are necessarily positive. A simple limiting procedure allows us to obtain solutions of (0.2).

1. **THE SUB-LINEAR CASE**

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary and \( f : [0, + \infty) \to \mathbb{R} \) be a \( C^1 \)-function with \( f(0) < 0 \). To study the problem (0.2) we consider the existence of non-zero solutions of the boundary value problem,

\[
-\Delta u(x) \in \mathcal{F}(u(x)) \quad a.e. \quad x \in \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( \mathcal{F} \) is the multivalued function defined by:

\[
\mathcal{F}(t) = \begin{cases} 
0, & \text{if } t < 0; \\
[f(0), 0], & \text{if } t = 0; \\
f(t), & \text{if } t > 0.
\end{cases}
\]

By a solution of (1.1) we mean a function \( u \in C^1(\overline{\Omega}) \cap C^2(\Omega^*) \) with \( \Omega^* = \{x \in \Omega / u(x) \neq 0\} \) and verifying (1.1). (Observe that \( \Delta u(x) \) is well-defined in \( \Omega^* \cap (\Omega - \overline{\Omega^*}) \)).

Notice that all solutions \( u \) of (1.1) are non-negative by the maximum principle; so they are solutions of (0.2). However, in contrast with [2] (where the case \( f(0) \geq 0 \) is studied) we cannot deduce that \( u > 0 \) in \( \Omega \).

In this section we will assume:

\[ (f_1) \quad f(0) < 0 \quad \text{and there exists } \theta > 0 \quad \text{such that } \quad f(\theta) = 0, \quad \text{with } f \text{ increasing in } [0, \theta]. \]

\[ (f_2) \quad f(s) \leq \alpha s + \beta, \quad \text{with } \beta \in \mathbb{R}, \quad 0 \leq \alpha < \lambda_1, \quad \text{where } \lambda_1 \text{ denotes the first eigenvalue of } -\Delta \text{ on } \Omega \text{ with zero Dirichlet boundary conditions.} \]

We will take a positive eigenfunction \( \varphi_1 \) associated to \( \lambda_1 \) such that: \( \|\varphi_1\|_{L^2(\Omega)} = 1 \). Let \( \sigma = \|\varphi_1\|_{L^2(\Omega)} \|\Omega\|^{1/2} \). (We denote \( |\Omega| \equiv \text{meas } \Omega \).

\[ (f_3) \quad \text{There exists } s_0, s_1, \gamma \in \mathbb{R} \text{ such that} \]

\[ i) \quad \theta < s_0 < s_1, \gamma > [\lambda_1 s_1^2 - 2f(0) \theta \sigma^2](s_1^2 - s_0^2 \sigma^2)^{-1} \equiv \gamma^*; \]

\[ ii) \quad f(s) \geq 0 \quad \forall s \in (\theta, s_0) \quad \text{and} \quad f(s) \geq \gamma s \quad \forall s \in (s_0, s_1). \]

**REMARKS 1.1. a)** A sufficient condition to assumption \( i) \) of \( (f_3) \) is

\[ i') \quad \theta < s_0 < (s_1^2 - 1)^{1/2} \sigma^{-1}, \gamma > \lambda_1 s_1^2 - 2f(0) \theta \sigma^2. \]

\[ b) \quad \text{Notice that } \gamma^* > \lambda_1, \text{ hence the meaning of } (f_3) \text{ is, roughly, that } f(s) \gg \lambda_1 s \text{ on a suitable interval } (s_0, s_1). \]

To study (1.1) we consider the sequence of problems:

\[
-\Delta u = f_n(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega
\]

where \( f_n : \mathbb{R} \to \mathbb{R} \) is of class \( C^1 \) and verifies \( f_n(t) = f(t) \quad \forall t > 1/n, \quad f_n(t) = 0 \quad \forall t \leq 0, \quad f_n(t) \geq f(t) \quad \forall t \in (0, 1/n), \) for all \( n \in \mathbb{N} \).
Changing $\beta$, if it is necessary, we can suppose the next uniform estimate for all $f_n$:

\begin{equation}
(1.3) \quad f_n(s) \leq as + \beta: \quad \forall s \geq 0 \text{ and } 0 < \beta.
\end{equation}

**Proposition 1.2.** Let us assume $(f_{1-3})$. For all $n \in \mathbb{N}$ the problem (1.2) has at least two nontrivial classical solutions $u_n \neq v_n$ verifying:

i) $u_n(x), v_n(x) > 0, \forall x \in \Omega$. ii) $\|u_n\|_\infty, \|v_n\|_\infty > \theta$.

**Proof.** Let $E = H^1(\Omega)$ be the usual Sobolev space, \[ \left( \text{with } \|u\|_E = \left( \int_\Omega |\nabla u(x)|^2 \, dx \right)^{1/2} \right) \]

We define the $C^1$-functionals $I_n : E \to \mathbb{R}$ by setting:

\[ I_n(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega F_n(u) \, dx, \quad \forall u \in E, \]

where $F_n(t) = \int_0^t f_n(s) \, ds$.

It is well-known that the critical points of $I_n$ are classical solutions of (1.2) and that (1.3) implies that $I_n$ is coercive and verifies the Palais-Smale condition [3]. Because of this, $I_n$ attains its infimum on a function $u_n$. Moreover, since $f_n'(0) = 0$, $I_n$ has a local minimum at 0.

On the other hand, let $\phi = s_1 \phi_1(\|\phi\|_\infty)^{-1}$.

By (f_3),

1) $\|\phi\|_{L^2(\Omega)} > \theta |\Omega|^{1/2}$;

2) $\theta < s_0 < \|\phi\|_{L^2(\Omega)} |\Omega|^{-1/2}$ and

\[ \gamma > \left[ \frac{\lambda_1}{\Omega} \int_\Omega \phi^2(x) \, dx - 2f(0) \theta |\Omega| \right] \left[ \int_\Omega \phi^2(x) \, dx - s_0^2 |\Omega| \right]^{-1}; \]

3) $f(s) \geq 0 \forall s \in (\theta, \|\phi\|_\infty)$ and $f(s) \geq \gamma s \forall s \in (s_0, \|\phi\|_\infty)$.

By 3),

\begin{equation}
(1.4) \quad F_n(s) \geq \int_0^s f(t) \, dt + \frac{\gamma}{2} (s^2 - s_0^2), \quad \forall s \in (s_0, \|\phi\|_\infty).
\end{equation}

Let $\Omega' = \{x \in \Omega/\phi(x) \geq s_0\}$. Notice that $\Omega' \neq \emptyset$, otherwise, $\phi(x) < s_0 \forall x \in \Omega$ implies $\|\phi\|_{L^2(\Omega)} < s_0 |\Omega|^{1/2}$, in contradiction with 1).

By (1.4), we have:

\[ \int_{\Omega'} F_n(\phi(x)) \, dx \geq \int_{\Omega'} \left( \phi^2(x) - s_0^2 \right) \, dx + \left( \int_0^{\phi(0)} f(t) \, dt \right) |\Omega'|. \]

Moreover,

\[ \int_{\Omega-\Omega'} F_n(\phi(x)) \, dx \geq \int_{\Omega-\Omega'} \left( \int_0^{\phi(x)} f(t) \, dt \right) \, dx \geq \left( \int_0^{\phi(0)} f(t) \, dt \right) |\Omega - \Omega'|. \]
Hence,
\[ I_n(\phi) = \frac{\lambda_1}{2} \int_\Omega \phi^2(x) \, dx - \int_\Omega F_n(\phi(x)) \, dx \leq \]
\[ \leq \frac{\lambda_1}{2} \int_\Omega \phi^2(x) \, dx - \frac{\gamma}{2} \int_\Omega (\phi^2(x) - s_0^2) \, dx - \left( \int_0^\delta f(t) \, dt \right) |\Omega| \leq \]
\[ \leq \frac{\lambda_1}{2} \int_\Omega \phi^2(x) \, dx - \frac{\gamma}{2} \int_\Omega (\phi^2(x) - s_0^2) \, dx - \left( \int_0^\delta f(t) \, dt \right) |\Omega| = \varepsilon_0. \]

By 2), it follows that \( I_n(\phi) \leq \varepsilon_0 < 0 \). So, all hypotheses of Mountain Pass Theorem [3], are verified and \( I_n \) has another critical point \( v_n \). In addition, there results
\[ (1.5) \quad I_n(u_n) \leq \varepsilon_0 < 0 < I_n(v_n) \]
which implies that \( u_n \neq v_n \), are non zero solutions of (1.2). Finally, simple applications of minimum and maximum principles imply i) and ii).  

In order to obtain solutions of (1.1), we need the next lemma:

**Lemma 1.3.** Under the hypotheses \((f_1-3)\), the sequences \( \{u_n\}, \{v_n\} \) have subsequences \( \{u_{n_k}\}, \{v_{n_k}\} \) such that \( \{u_n\} \to u_0, \{v_n\} \to v_0 \) in \( C^{1+\nu}(\overline{\Omega}) \), with \( 0 < \nu < 1 \).

**Proof.** By (1.3), we obtain an a priori estimate \( \|u_n\|_E, \|v_n\|_E \leq \beta |\Omega|^{1/2} (1 - \alpha/\lambda_1)^{-1}; \forall n \in \mathbb{N}, \) and using usual bootstrap arguments we obtain converging subsequences of \( \{u_n\}, \{v_n\} \) in \( C^{1+\nu}(\overline{\Omega}) \).  

From now on, we denote \( \{u_{n_k}\} = \{u_n\} \) and \( \{v_{n_k}\} = \{v_n\} \).

**Theorem 1.4.** Let us assume \((f_1-3)\). There exist at least two distinct, non-negative and non-zero solutions of (1.1).

**Proof.** Let \( u_0, v_0 \) be given by Lemma 1.3. Clearly, \( u_0, v_0 \) are non-negative and non-zero by Proposition 1.2-ii) and Lemma 1.3.

Notice that \( \lim_{n \to \infty} F_n(t) = F(t) \quad \forall t \in \mathbb{R} \), where
\[ F(t) = \begin{cases} 0, & \text{if } t < 0; \\ \int_0^t f(s) \, ds, & \text{if } t \geq 0. \end{cases} \]

So, by Lemma 1.3 and the Lebesgue’s dominated convergence theorem:
\[ \lim_{n \to \infty} \int_\Omega |\nabla u_n(x)|^2 \, dx = \int_\Omega |\nabla u_0(x)|^2 \, dx \]
and
\[ \lim_{n \to \infty} \int_\Omega F_n(u_n(x)) \, dx = \int_\Omega F(u_0(x)) \, dx. \]

Then \( \lim_{n \to \infty} I_n(u_n) = I(u_0) \), where
\[ I(u) : = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega F(u(x)) \, dx. \]
Similar arguments prove \( \lim_{n \to \infty} I_n(v_n) = I(v_0) \). Hence, by (1.5) \( I(u_0) \leq \delta_0 < 0 \leq I(v_0) \) and \( u_0 \neq v_0 \).

In order to prove that \( u_0, v_0 \) are solutions of (1.1), we observe that

\[
\lim_{n \to \infty} f_n(x_n) = f(x) \text{ if } \lim_{n \to \infty} x_n = x > 0.
\]

Then, if \( \Omega^* = \{ x \in \Omega / u_0(x) \neq 0 \} \),

\[
\lim_{n \to \infty} f_n(u_n(x)) = f(u_0(x)) \quad \forall x \in \Omega^*.
\]

So that, for all \( u \in C_0^\infty(\Omega^*) \), the equalities

\[
\int_{\Omega^*} \nabla u_n(x) \nabla u(x) \, dx + \int_{\Omega^*} f_n(u_n(x)) u(x) \, dx = 0
\]

and Lemma 1.3 imply

\[
\int_{\Omega^*} \nabla u_0(x) \nabla u(x) \, dx + \int_{\Omega^*} f(u_0(x)) u(x) \, dx = 0.
\]

In particular, \( u_0 \in C^2(\Omega^*) \) and \(-\Delta u_0(x) = f(u(x)), \text{ in } \Omega^*\).

Finally, by a Morrey-Stampacchia theorem (see [9, Theorem 3.2.2, p. 69]), we have also \(-\Delta u_0(x) = 0 \text{ a.e. } \Omega - \Omega^*\), and so \( u_0 \) is a solution of (1.1).

The same ideas show \( v_0 \) is another solution of (1.1).

**Remark 1.5.** Observe that our technique can be combined with some symmetry properties of the domain. More precisely, if \( \Omega \) is symmetric in the sense of Steiner [8] (i.e. \( \Omega \) is symmetric with respect to a plane, for instance \( x_i = 0 \), and convex in the variable \( x_i \)), we deduce [7] that \( u_n, v_n \) are symmetric (in the sense of Steiner). Hence, their limits \( u_0, v_0 \) (which are solutions of (1.1) as it has been proved) are symmetric also.

### 2. The superlinear case

Our method for study (1.1) can be useful to prove existence of solutions for other hypotheses on \( f \). For instance, the superlinear case. We assume:

\begin{enumerate}
  \item[(f_4)] There exist \( a_1, a_2 \geq 0 \) such that
    \[
    |f(s)| \leq \begin{cases} 
    a_1 + a_2|s|^\mu, & \text{if } N > 2, \\
    a_1 \exp(\phi(s)), & \text{if } N = 2,
    \end{cases}
    \]
    where \( 0 < \mu < (N + 2)(N - 2)^{-2} \).
  \item[(f_5)] There exist \( \rho > 2 \) and \( r \geq 0 \) such that \( 0 < \rho F(s) \leq s f(s) \quad \forall s \geq r. \)
\end{enumerate}

**Theorem 2.1.** Let us assume \((f_1), (f_4-5)\). Then, the problem (1.1) has at least one non-negative and nonzero solution.

**Proof.** Let \( f_n, F_n, F, I_n \) and \( I \) be functions like in section 1. Since by \((f_4)\) (see [3])

\[
\lim_{t \to \infty} I(t\phi) = -\infty,
\]

we deduce that there exists \( t_0 > 0 \) such that \( I_n(t_0 \phi_1) \leq I(t_0 \phi_1) < 0 \) for all \( n \in \mathbb{N} \).

Moreover, \( f_n \) satisfies \((f_5)\) and \( f_n'(0) = 0 \). Then \( I_n \) verifies all hypotheses of the
Mountain Pass Theorem [3] with \( \bar{c} = t_0\tilde{\phi}_0 \) (independently of \( n \in \mathbb{N} \)). Consequently, it has a critical point \( u_n \) such that
\[
0 < I_n(u_n) \leq \max_{\bar{e} \in (0,1)} I_n(\bar{e}) \leq \max_{\bar{e} \in (0,1)} I(\bar{e}) \equiv M, \quad u_n > 0 \text{ in } \Omega, \quad \|u_n\|_{\infty} > \theta,
\]
for all \( n \in \mathbb{N} \).

In order to prove an a priori estimate of \( u_n \), observe that
\[
M \geq I_n(u_n) = I_n(u_n) - \frac{1}{\rho} I'_n(u_n) u_n = \left( \frac{1}{2} - \frac{1}{\rho} \right) \|u_n\|_{E}^2 + \int_{\partial} \left[ \frac{f_n(u_n)}{\rho} u_n - F_n(u_n) \right] dx.
\]
Since \( f_n \geq f \) and \( |F_n(u) - F(u)| \leq -f(0) u \ \forall u \geq 0 \), we deduce
\[
M \geq \left( \frac{1}{2} - \frac{1}{\rho} \right) \|u_n\|_{E}^2 + \int_{\partial} \left[ \frac{f_n(u_n)}{\rho} u_n - F(u_n) \right] dx + f(0) \int_{\partial} u_n dx.
\]
By \((f_3)\), it follows \( M \geq (1/2 - 1/\rho)\|u_n\|_{E}^2 - k\|u_n\|_{E} - c \) where \( k, c > 0 \), and hence \( \{u_n\} \) is bounded in \( E \).

Then a limiting procedure and similar arguments to those of Theorem 1.4 conclude the proof.

REMARKS 2.2. a) Notice that from the results of [5] if, in addition to \((f_1), (f_4-5)\), we assume that \( f \) is non-increasing, we can consider a «small» ball \( B \subset \Omega \) and a positive solution \( u \) of \(-\Delta u = f(u) \) in \( B \), \( u = 0 \) on \( \partial B \).

Hence, the extension zero in \( \Omega - B \) yields a weak solution \( u \) of \(-\Delta u \in \mathcal{F}(u) \) a.e. in \( \Omega \), \( u = 0 \) on \( \partial \Omega \).

We remark that this procedure gives us solutions which may be not of class \( C^1 \) (see [5, Theorem 1.1]).

b) Observe also that in the case that \( \Omega = B(r) \) is the ball of radius \( r \) centered at \( x = 0 \) and under the conditions of \( a) \), we know by [4] that there exists \( r^* > 0 \) such that the problem \(-\Delta u = f(u) \) in \( B(r) \), \( u = 0 \) on \( \partial B \) has no radial nonnegative solutions for \( r > r^* \).

On the other side, we have obtained, see Theorem 2.1 and Remark 1.5, the existence of one \( C^1 \)-radial non-negative solution of the multivalued problem (1.1), for all \( r > 0 \). This proves the existence of an \( r_0 \) and a positive radial solution \( u \) of \(-\Delta u = f(u) \) in \( B(r_0) \), \( u = 0 \) on \( \partial B(r_0) \) such that \( u'(r_0) = 0 \) (we denote \( u(s) = u(|x|), x \in B(r) \)). This kind of solutions differ from those of [5] where \( u'(r_0) < 0 \).

ACKNOWLEDGEMENTS

The authors wish to thank A. Ambrosetti for fruitful discussions and suggestions. Also, the first author (supported by a grant of the Junta de Andalucia and Univ. of Granada, Spain) is grateful for the kind hospitality from the Scuola Normale Superiore di Pisa while this paper was written. The second author is supported by Italian Min. P.I. (40%).
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