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Adrian Constantin

A Gronwall-like inequality and its applications

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Analisi matematica. — *A Gronwall-like inequality and its applications.* Nota di

ADRIAN CONSTANTIN, presentata (*) dal Corrisp. R. CONTI.

ABSTRACT. — A generalized Gronwall-like inequality is established and applied in obtaining a right saturated solution for a class of differential equations and in estimating the solution of an evolution equation for the so called hidden variables.

KEY WORDS: Gronwall inequality; Hidden variables; Integral equations.

RIASSUNTO. — *Una disuguaglianza del tipo di Gronwall e sue applicazioni.* Si stabilisce una disuguaglianza di tipo Gronwall che si applica a una classe di equazioni differenziali ed a una stima di soluzioni delle equazioni di evoluzione con variabili latenti.

1. A. Morro [3] obtained interesting new results on the mathematical structure of the hidden variable model with applications to continuum thermodynamics. A Gronwall-like inequality with applications to the evolution equation for hidden variables and new insights into the asymptotic stability have been given. It is the aim of this note to establish a Gronwall-like generalized inequality and to apply it to a certain class of differential equations and to an estimate of the solution of the evolution equation for hidden variables.

2. Proving the existence of solutions of ordinary differential equations it is usual to get some integral inequalities. In connection with this we give the following:

THEOREM. Let g, h, k be real continuous functions on the interval $[t_0, t_1]$, $t_0, t_1 \in \mathbb{R}$, $t_1 > t_0$; $g > 0$, $kh > 0$ and $k, g \in \mathcal{C}^1([t_0, t_1], \mathbb{R})$, $w : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous, monotone non-decreasing function so that exists $L > 0$ with $w(x) \geq Lx$ for each $x \in \mathbb{R}_+$ and $w(0) = 0$. If a continuous, positive function v has the property that

$$v(t) \leq g(t) + k(t) \int_{t_0}^{t} h(s)\, w(v(s))\, ds \qquad t \in [t_0, t_1] \tag{1}$$

then

$$v(t) \leq \Phi^{-1}(\Phi(g(t_0))) + \int_{t_0}^{t} [k(s)\, h(s) + \max\{0, k'(s)/Lk(s)\} + \max\{0, g'(s)/Lg(s)\}]\, ds \tag{2}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is the function $\Phi(u) = \int_{u_0}^{u} w(s)^{-1}\, ds$, $u_0 \neq 0$.

PROOF. We denote by y the function

$$y(t) = \int_{t_0}^{t} h(s)\, w(v(s))\, ds$$

and from (1) we obtain that $v(t) \leq g(t) + k(t)\, y(t)$ so that

$$k(t)\, y'(t) = k(t)\, h(t)\, w(v(t)) \leq k(t)\, h(t)\, w[g(t) + k(t)\, y(t)] \tag{3}$$

(*) Nella seduta del 9 dicembre 1989.

which implies that $k(t)\,y'(t)/w[g(t)+k(t)\,y(t)] \leqslant k(t)\,h(t)$ since $kh>0$ and w is monotone non-decreasing.

We observe that because $kh>0$ we have $ky>0$ for $t>t_0$ and so

$$g'(t) \leqslant 0 \Rightarrow g'(t)/w[g(t)+k(t)\,y(t)] \leqslant 0\,,$$

$$g'(t) > 0 \Rightarrow g'(t)/w[g(t)+k(t)\,y(t)] \leqslant g'(t)/L[g(t)+k(t)\,y(t)] \leqslant g'(t)/Lg(t)\,.$$

Hence we have that

$$g'(t)/w[g(t)+k(t)\,y(t)] \leqslant \max\{0, g'(s)/Lg(s)\}\,. \tag{4}$$

It is now obvious that for $t>t_0$

$$k'(t) \leqslant 0\,,\quad y(t)>0 \Rightarrow y(t)\,k'(t)/w[g(t)+k(t)\,y(t)] \leqslant 0\,,$$

$$\begin{aligned} k'(t) > 0\,,\quad y(t)>0 \Rightarrow{} & y(t)\,k'(t)/w[g(t)+k(t)\,y(t)] \leqslant \\ & \leqslant y(t)\,k'(t)/L[g(t)+k(t)\,y(t)] \leqslant y(t)\,k'(t)/Lk(t)\,y(t) = k'(t)/Lk(t)\,, \end{aligned}$$

$$k'(t) > 0\,,\quad y(t)<0 \Rightarrow y(t)\,k'(t)/w[g(t)+k(t)\,y(t)] \leqslant 0\,,$$

$$\begin{aligned} k'(t) \leqslant 0\,,\quad y(t)<0 \Rightarrow{} & k'(t)\,y(t)/w[g(t)+k(t)\,y(t)] \leqslant \\ & \leqslant k'(t)\,y(t)/L[g(t)+k(t)\,y(t)] \leqslant k'(t)\,y(t)/Lk(t)\,y(t) = k'(t)/Lk(t) \end{aligned}$$

and we obtain for $t>t_0$

$$y(t)\,k'(t)/w[g(t)+k(t)\,y(t)] \leqslant \max\{0, k'(t)/Lk(t)\}\,. \tag{5}$$

From (3), (4), (5) we conclude that

$$\begin{aligned} & [g'(t)+y'(t)\,k(t)+k'(t)\,y(t)]/w[g(t)+k(t)\,y(t)] \leqslant \\ & \qquad \leqslant k(t)\,h(t) + \max\{0, k'(t)/Lk(t)\} + \max\{0, k'(s)/Lk(s)\}\,. \end{aligned}$$

By integration on $[t_0, t]$ this yields

$$\int\limits_{g(t_0)+k(t_0)\,y(t_0)}^{g(t)+k(t)\,y(t)} w(s)^{-1}\,ds \leqslant \int\limits_{t_0}^{t} [k(s)\,h(s) + \max\{0, g'(s)/Lg(s)\} + \max\{0, k'(s)/Lk(s)\}]\,ds$$

and because $y(t_0)=0$ we deduced that

$$\begin{aligned} \Phi(g(t)+k(t)\,y(t)) - \Phi(g(t_0)) \leqslant & \int\limits_{t_0}^{t} [k(s)\,h(s) + \\ & \qquad + \max\{0, g'(s)/Lg(s)\} + \max\{0, k'(s)/Lk(s)\}]\,ds \end{aligned}$$

from which

$$\begin{aligned} g(t)+k(t)\,y(t) \leqslant{} & \Phi^{-1}(\Phi(g(t_0))) + \\ & + \int\limits_{t_0}^{t} [k(s)\,h(s) + \max\{0, g'(s)/Lg(s)\} + \max\{0, k'(s)/Lk(s)\}]\,ds \end{aligned}$$

and since $v(t) \leqslant g(t)+k(t)\,y(t)$ the theorem is proved.

3. We consider the differential equation

$$v'(t) = a(t)\,w(v(t)) + b(t) \qquad t\in[t_0, +\infty),\ t_0>0$$

where $a(t), b(t) : [t_0, \infty) \to \mathbb{R}_+$ are two continuous functions and $w : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz with the properties from the theorem.

We will show that for every $x_0 > 0$ this equation has a unique solution v on $[t_0, \infty)$ with the initial condition $v(t_0) = x_0$.

This equation can be written under the form $x'(t) = f(t,x)$ where $f(t,x) = a(t)w(x) + b(t)$. Let us show that the function f is locally Lipschitz on $[t_0, \infty) \times [0, \infty)$. We have that for every $t', x' \in [t_0, \infty) \times [0, \infty)$ there is a neighborhood $V_{t'}$ of t' and an $M_{V_{t'}} > 0$ so that $|a(t)| \leqslant M_{V_{t'}}$ for each $t \in V_{t'} \cap [t_0, \infty)$ because $a(t)$ is continuous on $[t_0, \infty)$ and there is a neighborhood $V_{x'}$ of x' and an $M_{V_{x'}}$ so that $|w(x) - w(y)| \leqslant M_{V_{x'}}|x - y|$, $(\forall)\, x, y \in V_{x'} \cap [0, \infty)$, because w is locally Lipschitz on $[0, \infty)$. We deduce that on $V_{t'} \times V_{x'}$ we have $|f(t,x) - f(t,y)| = |a(t)|\,|w(x) - w(y)| \leqslant M_{V_{t'}} \cdot M_{V_{x'}}|x - y|$, $(\forall)(t,x), (t,y) \in V_{t'} \times V_{x'}$ and so f is locally Lipschitz and continuous.

Applying the existence and unicity theorem we deduce that there is a right-saturated solution v with the initial condition $v(t_0) = x_0$ defined on $[t_0, T)$. It is known that if v is a right-saturated solution defined on $[t_0, T)$ then there are only two possibilities: $T = +\infty$ or $\lim_{t \to T}|v(t)| = \infty$ (the second case is known in the literature as «blow-up»).

We have that for this solution $v'(t) \geqslant 0$ for every $t \in [t_0, T)$ and so $v(t) > 0$ for every $t \in [t_0, T)$ because $v(t_0) = x_0 > 0$.

We observe that $v'(t) = t[a(t)/t]\, w(v(t)) + b(t)$ from which we obtain the inequality

$$v'(t) \leqslant \int_{t_0}^{t} [a(s)/s]\, w(v(s))\, ds + t[a(t)/t]\, w(v(t)) + b(t)$$

and an integration yields

$$v(t) \leqslant x_0 + \int_{t_0}^{t} b(s)\, ds + t \int_{t_0}^{t} [a(s)/s]\, w(v(s))\, ds\,.$$

Let us assume:

$$g(t) = x_0 + \int_{t_0}^{t} b(s)\, ds\,, \qquad k(t) = t\,, \qquad h(t) = a(t)/t\,.$$

Applying the proved theorem we obtain that

$$v(t) \leqslant \Phi^{-1}\left(\Phi(x_0) + \int_{t_0}^{t} a(s)\, ds + L^{-1}\ln t - L^{-1}\ln t_0 + L^{-1}\ln g(t) - L^{-1}\ln x_0\right) \leqslant$$

$$\leqslant \Phi^{-1}\left(\Phi(x_0) + \int_{t_0}^{T} a(s) ds + L^{-1}\ln T - L^{-1}\ln t_0 + L^{-1}\ln g(T) - L^{-1}\ln x_0\right) < c, \quad (\forall)\ t \in [t_0, T)$$

where c is a positive constant.

Hence the only possibility is that $T = \infty$.

4. We now recall from [3] some notions on materials with hidden variables and we give a similar estimate of solution of the evolution equation for the hidden variables under a slightly generalized hypothesis.

A material with hidden variables $\{y_0, z_0, a_0, \mathrm{U}, V, \mathcal{C}, f\}$ on $Y \times Z \times A$ consists of a ground value (y_0, z_0, a_0) of the variables $(y, z, a) \in Y \times Z \times A$, a representing the set of hidden variables, together with a connected neighborhood $U \times V$ of (y_0, z_0) and the maps $\mathcal{C} \in \mathcal{C}^2(U \times A, \Phi)$, $f \in \mathcal{C}^2(U \times V \times A, A)$ where V, Z, A and Φ denote finite dimensional real normed vector spaces with $\dim A \leqslant \dim Y + \dim Z$.

A path is a bounded and piecewise continuously differentiable map $\pi : \mathbb{R} \to U \times V$. The hidden variables are functions on $\mathbb{R}$ (time).

Consider the evolution equation $\dot{a}(t) = f(\pi(t), a(t))$, $t \geqslant t_0$, $a(t_0) = a^*$ where the evolution function f satisfies

I) there is a map $\Lambda \in \mathcal{L}(A, A)$ and a function w like that in the theorem such that

$$\|f(\pi, a+b) - f(\pi, a) - \Lambda b\| \leqslant w(\|b\|), \qquad \pi \in U \times V,\ a, a+b \in A$$

and each eigenvalue of $\Lambda + I_A$ has a negative real part,

II) there is a positive constant ε such that

$$\|f(\pi + \omega, a) - f(\pi, a)\| \leqslant \varepsilon \|\omega\| \qquad \pi, \pi + \omega \in U \times V,\ a \in A.$$

We observe that this class of evolution equations is larger than that considered by A. Morro [3] because it contains all functions of the form $w(\|b\|) = \delta \|b\|$ and also others (for example $w(\|b\|) = \|b\| + \|b\|^2$). It should be noted, however, that in [3] it is not necessary to assume $g > 0$.

Let as in [3] the hidden variables be a, $a + b \in A$ corresponding to the paths π, $\pi + \omega$. We have

$$\dot{a}(t) = f(\pi(t), a(t)) \qquad t \geqslant t_0,\ a(t_0) = a^*$$

$$\dot{a}(t) + \dot{b}(t) = f(\pi(t) + \omega(b), a(t) + b(t)), \qquad t \geqslant t_0,\ a(t_0) + b(t_0) = a^* + b^*.$$

Denoting $\gamma = f(\pi + \omega, a + b) - f(\pi, a + b)$, $r = f(\pi, a + b) - f(\pi, a) - \Lambda b$ we obtain that $\dot{b} = \gamma + r + \Lambda b$.

We find that $(d/dt)[\exp(-t\Lambda)\, b(t)] = \exp(-t\Lambda)[\gamma(t) + r(t)]$ and an integration yields

$$b(t) = \exp((t - t_0)\Lambda)\, b(t_0) + \int_{t_0}^{t} \exp((t - s)\Lambda)[\gamma(s) + r(s)]\, ds.$$

Considering I) and II) and denoting by $-m$ the real part of the eigenvalues of Λ with the greatest real part, it follows that

$$\|b(t)\| \leqslant \exp(-m(t - t_0))\|b(t_0)\| + \int_{t_0}^{t} \exp(-m(t - s))\|\gamma(s) + r(s)\|\, ds \leqslant$$

$$\leqslant \exp(-m(t - t_0))\|b(t_0)\| + \varepsilon \int_{t_0}^{t} \exp(-m(t - s))\|\omega(s)\|\, ds + \int_{t_0}^{t} \exp(-m(t - s))\, w(\|b(s)\|)\, ds.$$

Let us assume: $v(t) = \|b(t)\|$, $\quad k(t) = \exp(-mt)$, $\quad h(t) = \exp(mt)$,

$$g(t) = \exp(-m(t - t_0))\|b(t_0)\| + \varepsilon \int_{t_0}^{t} \exp(-m(t - s))\|\omega(s)\|\, ds.$$

From the previous theorem we obtain that

$$v(t) \leqslant \Phi^{-1}\left(\Phi(\|b(t_0)\|) + t - t_0 + \int_{t_0}^{t} \max\{0, g'(s)/Lg(s)\}\, ds \right).$$

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Universitatea din Timişoara
Bulev. V. Pârvan Nr. 4
Timişoara 1900 (Romania)