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A continuous version of the Filippov-Gronwall inequality for differential inclusions

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Analisi matematica. — *A continuous version of the Filippov-Gronwall inequality for differential inclusions.* Nota (*) di ANTÓNIO ORNELAS, presentata dal Corrisp. R. CONTI.

ABSTRACT. — We give an estimate for the distance between a given approximate solution for a Lipschitz differential inclusion and a true solution, both depending continuously on initial data.

KEY WORDS: Differential inclusions; Filippov-Gronwall inequality; Relaxed solutions.

RIASSUNTO. — *Una versione continua della disuguaglianza di Filippov-Gronwall per inclusioni differenziali.* Si determina una stima per la distanza fra una data soluzione approssimata di una inclusione differenziale lipschitziana e una vera soluzione, con dipendenza continua dai dati iniziali.

1. INTRODUCTION

Consider the Cauchy problem

(F) $$x'(t) \in F(t,x), \qquad x(0) = \xi,$$

where $F: I \times \mathbb{R}^n \to \mathbb{R}^n$ is a multifunction Lipschitz in x with closed values, without any convexity or boundedness assumption. Filippov [5] proved that a solution x of (F) exists iff there exists an approximate solution y, *i.e.* an absolutely continuous $y: I \to \mathbb{R}^n$ such that the distance $\rho(t) := d[y'(t), F(t, y(t))]$ is integrable. Moreover he showed that the distance from the given approximate solution y to a true solution x verifies the estimate:

$$\int_0^t |y'(\tau) - x'(\tau)|\, d\tau \leq \int_0^t \exp\left[\int_\tau^t k(r)\, dr\right] \rho(\tau)\, d\tau,$$

where $k(t)$ is the Lipschitz constant of $F(t,\cdot)$. It is easy to build simple examples in which for most values of initial data ξ there exists a unique «Filippov solution», *i.e.* a true solution x which is the unique solution of (F) verifying the above estimate. However this solution changes discontinuously with data at isolated points where uniqueness lacks, showing that to obtain a solution depending continuously on data one must relax the estimate.

Aim of this paper is to let y' vary continuously in L^1 with the parameter ξ over a compact Ξ in $\mathbb{R}^n$ and to construct a solution $x(\xi)$ verifying Filippov's estimate plus an arbitrarily small error δ and such that x and x' depend continuously on ξ.

Himmelberg-Van Vleck [8] extended Filippov's result from a jointly continuous to a jointly measurable multifunction F. Solutions depending continuously on ξ were also obtained in [3] and [4], starting from rather special approximate solutions. For more information on differential inclusions like (F), see [2].

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2. Assumptions

We say that $F: A \to B$ is a multifunction if F is a map associating to each point $a \in A$ a nonempty subset $F(a)$ of B. We denote by $dl(A, B)$ the Hausdorff «distance» between two sets A, B in $\mathbb{R}^n$, generated by the euclidian norm $|\cdot|$, as in [8]. The distance $d(x, A)$ from a point x to a set A is $\inf\{|x-a| : a \in A\}$. It is well-known that $d(x, B) \leq d(x, A) + dl(A, B)$ and $|d(x, A) - d(y, B)| \leq |x - y| + dl(A, B)$. I is the interval $[0, T]$, and χ_E is the characteristic function of a subset E of I. We consider the Banach space AC of absolutely continuous maps $x: I \to \mathbb{R}^n$ with norm

$$|x|_{AC} := |x(0)| + \int_0^T |x'(\tau)|\, d\tau,$$

and assume that Ξ is a compact subset of $\mathbb{R}^n$, and:

Hypothesis (H). $F: I \times \mathbb{R}^n \to \mathbb{R}^n$ is a multifunction with:

a) values $F(t, x)$ closed;

b) $t \mapsto F(t, x)$ Lebesgue measurable (see [7]);

c) $dl\{F(t,x), F(t, \underline{x})\} \leq k(t)|x - \underline{x}|$ *a.e.* on I, for some $k \in L^1(I)$;

d) $t \mapsto d[y'(t),\ F(t, y(t))]$ is in $L^1(I)$ for some y in AC.

3. Result

Theorem. *Let F satisfy hypothesis* (H) *and fix $\delta > 0$ and a continuous map $y: \Xi \to$ AC. Then there exists a continuous map $x: \Xi \to$ AC verifying*:

$$x(\xi)'(t) \in F[t, x(\xi)(t)] \qquad a.e. \text{ on } I,$$

$$\int_0^t |x(\xi)'(\tau) - y(\xi)(\tau)|\, d\tau \leq \delta + \int_0^t \exp\left[\int_\tau^t k(r)\, dr\right] \rho(\xi)(\tau)\, d\tau,$$

where $\rho(\xi)(t) := d[y(\xi)'(t),\ F(t, y(\xi)(t))]$.

Proof. The proof is essentially Filippov's construction of successive approximations. As a function of initial data ξ each approximation would not be continuous so we modify it, in order to obtain continuity, by interpolating with continuous partitions of the interval, as in [1]. However, unlike in [4], we need Fryszkowski's version of the Liapunov theorem on the range of a vector measure, since $y(\xi)'(t)$ now depends on ξ and we aim at a sharper result.

a) Set $\varepsilon := \delta \exp[-2m(t)]/6$, where

$$m(t) := \int_0^t k(\tau)\, d\tau.$$

Consider the map $\rho: \Xi \to L^1(I)$, with $\rho(\xi)(t)$ as in the statement; using (*d*) of hypothesis (H) one easily sees that ρ is well-defined. Moreover ρ is continuous because

$$\int |\rho(\xi) - \rho(\underline{\xi})|\, d\tau \leq \int |y(\xi)' - y(\underline{\xi})'|\, d\tau + \int k(\tau)|y(\xi) - y(\underline{\xi})|\, d\tau \leq |y(\xi) - y(\underline{\xi})|_{AC}[1 + m(T)].$$

For each $\xi_j \in \Xi$, the maps $\beta_j, \sigma_j : \Xi \to L^1(I)$, $\beta_j(\xi)(t) := |y(\xi_j)'(t) - y(\xi)'(t)|$, $\sigma_j(\xi)(t) := |\rho(\xi_j)(t) - \rho(\xi)(t)|$, are continuous and verify $\beta_j(\xi_j) = \sigma_j(\xi_j) = 0$; hence the set

$$\Xi_j := \left\{\xi \in \Xi : |y(\xi) - y(\xi_j)|_{AC} < \varepsilon/2\,,\ \int \sigma_j(\xi)\, d\tau < \varepsilon/2\right\}$$

is an open nbd of ξ_j. Therefore there exists an open cover $\Xi_1, \dots, \Xi_m$ of Ξ and a subordinate continuous partition of unity $\pi_1, \dots, \pi_m : \Xi \to [0,1]$. Choose $\Delta > 0$ such that

$$\text{measure}\,(E) \leqslant \Delta \Rightarrow \int_E \rho_1(\tau)\, d\tau < \varepsilon/2\,,$$

where $\rho_1 \in L^1(I)$ is such that $\rho(\xi_j)(t) \leqslant \rho_1(t)$ *a.e.* on I for $j = 1, \dots, m$. For each $\xi \in \Xi$ find a map $v^0(\xi) \in L^1(I)$ such that $v^0(\xi)(t) \in F[t, y(\xi)(t)]$, $|v^0(\xi)(t) - y(\xi)'(t)| = \rho(\xi)(t)$ *a.e.* on I, using the measurable selection theorem of Kuratowski-Ryll Nardzewski (see [7]). Apply the technique of [1] or [6] to find sets $E_1(\xi), \dots, E_m(\xi)$ such that:

$$\text{measure}\,[E_j(\xi)] = T \cdot \pi_j(\xi)\,, \qquad \int_0^{l\Delta} \chi_{E_j(\xi)}\, \rho(\xi_j)\, d\tau \leqslant \pi_j(\xi) \int_0^{l\Delta} \rho(\xi_j)\, d\tau + \varepsilon/2m\,,$$

$$\int \chi_{E_j(\xi)}\, \beta_j(\xi)\, d\tau \leqslant \pi_j(\xi) \int \beta_j(\xi)\, d\tau + \varepsilon/2m \leqslant [\pi_j(\xi) + 1/m]\, \varepsilon/2\,.$$

Define

$$x^1(\xi)(t) := y(\xi)(0) + \int_0^t \sum_{j=1}^m \chi_{E_j(\xi)}\, v^0(\xi_j)\, d\tau\,,$$

obtaining a continuous map $x^1 : \Xi \to AC$ such that

$$\int_0^t |x^1(\xi)' - y(\xi)'|\, d\tau = \int_0^t \sum_{j=1}^m \chi_{E_j(\xi)} |v^0(\xi_j) - y(\xi)'|\, d\tau \leqslant$$

$$\leqslant \int_0^{l\Delta} \sum_j \chi_{E_j(\xi)}\, \rho(\xi_j)\, d\tau + \int \sum_j \chi_{E_j(\xi)}\, \beta_j(\xi)\, d\tau + \int_{l\Delta}^t \rho_1(\tau)\, d\tau \leqslant$$

$$\leqslant \sum_{j=1}^m \left[\pi_j(\xi) \int_0^{l\Delta} \rho(\xi_j)\, d\tau + \frac{\varepsilon}{2m}\right] + \sum_{j=1}^m \left[\pi_j(\xi) + \frac{1}{m}\right] \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leqslant$$

$$\leqslant \int_0^t \rho(\xi)\, d\tau + \sum_j \pi_j(\xi) \int \sigma_j(\xi)\, d\tau + 4\varepsilon/2 \leqslant \int_0^t \rho(\xi)\, d\tau + 5\varepsilon/2\,.$$

If we fix $t \in I$ and let j be such that $t \in E_j(\xi)$ then

$$d[x^1(\xi)'(t), F(t, y(\xi)(t))] = d[v^0(\xi_j)(t), F(t, y(\xi)(t))] \leqslant dl[F(t, y(\xi_j)(t)), F(t, y(\xi)(t))] \leqslant$$

$$\leqslant k(t) |y(\xi_j)(t) - y(\xi)(t)| \leqslant k(t) |y(\xi_j) - y(\xi)|_{AC} \leqslant k(t)\, \varepsilon/2\,,$$

hence:

$$d[x^1(\xi)'(t), F(t, x^1(\xi)(t))] \leqslant d[x^1(\xi)'(t), F(t, y(\xi)(t))] + dl[F(t, y(\xi)(t)), F(t, x^1(\xi)(t))] \leqslant$$

$$\leqslant k(t) \left[\int_0^t \rho(\xi)\, d\tau + 6\varepsilon/2\right].$$

b) Claim. For $n = 1, 2, \dots$, it is possible to define a continuous map $x^n: \Xi \to \mathrm{AC}$ verifying $x^n(\xi)(0) = y(\xi)(0)$ and:

(i) $$\int_0^t |x^n(\xi)' - x^{n-1}(\xi)'| \, d\tau \leq \int_0^t [m(t) - m(\tau)]^{n-1} [(n-1)!]^{-1} \rho(\xi)(\tau) \, d\tau + \\ + 5 \cdot 2^{-n} \varepsilon + 6 \cdot 2^{-n} \varepsilon \sum_{i=1}^{n-1} [2m(t)]^i / i! \, ;$$

(ii) $$d[x^n(\xi)'(t), F(t, x^n(\xi)(t))] \leq k(t) \int_0^t [m(t) - m(\tau)]^{n-1} [(n-1)!]^{-1} \rho(\xi)(\tau) \, d\tau + \\ + 6 \cdot 2^{-n} \varepsilon k(t) \sum_{i=0}^{n-1} [2m(t)]^i / i! \, .$$

To prove this claim notice first that for $n = 1$ it has been proved above. Supposing that it holds true for $n - 1$, we prove it now for n.

Consider the map $\rho^n: \Xi \to L^1(I)$, $\rho^n(\xi)(t) := d[x^{n-1}(\xi)'(t), F(t, x^{n-1}(\xi)(t))]$; since $x^{n-1}: \Xi \to \mathrm{AC}$ is continuous by hypothesis, reasoning as for ρ at the beginning of the proof one easily sees that ρ^n is well-defined and continuous.

Moreover the estimate (ii) of the claim for $n - 1$ gives

$$\int_0^t \rho^n(\xi) \, d\tau \leq \int_0^t k(\tau) \int_0^\tau [m(\tau) - m(r)]^{n-2} [(n-2)!]^{-1} \rho(\xi)(r) \, dr \, d\tau + \\ + 6 \cdot 2^{-n+1} \varepsilon \int_0^t k(\tau) \sum_{i=0}^{n-2} ([2m(\tau)]^i / i!) \, d\tau \leq \\ \leq \int_0^t [m(t) - m(\tau)]^{n-1} [(n-1)!]^{-1} \rho(\xi)(\tau) \, d\tau + 6 \cdot 2^{-n} \varepsilon \sum_{i=1}^{n-1} [2m(t)]^i / i! \, .$$

As for the case $n = 1$ above, it is possible to find an open cover $\Xi_1^n, \dots, \Xi_{m_n}^n$ of Ξ and a subordinate continuous partition of unity $\pi_1^n, \dots, \pi_{m_n}^n: \Xi \to [0, 1]$ such that each set Ξ_j^n is an open nbd of ξ_j^n given by:

$$\Xi_j^n := \left\{ \xi \in \Xi : |x^{n-1}(\xi) - x^{n-1}(\xi_j^n)|_{\mathrm{AC}} < 2^{-n} \varepsilon \, , \ \int \sigma_j^n(\xi) \, d\tau < 2^{-n} \varepsilon \right\},$$

where $\sigma_j^n(\xi)(t) := |\rho^n(\xi_j^n)(t) - \rho^n(\xi)(t)|$. Set $\beta_j^n(\xi)(t) := |x^{n-1}(\xi_j^n)'(t) - x^{n-1}(\xi)'(t)|$, and choose $\Delta^n > 0$ such that:

$$\text{measure}\,(E) \leq \Delta^n \Rightarrow \int_E \rho_n(\tau) \, d\tau < 2^{-n} \varepsilon \, ,$$

where ρ_n is some map in $L^1(I)$ such that $\rho^n(\xi_j^n)(t) \leq \rho_n(t)$ *a.e.* on I, $\forall j = 1, \dots, m_n$. Find a map $v^{n-1}(\xi)$ in L^1 such that $v^{n-1}(\xi)(t) \in F[t, x^{n-1}(\xi)(t)]$, $|v^{n-1}(\xi)(t) - x^{n-1}(\xi)'(t)| = \rho^n(\xi)(t)$ *a.e.* on I, and apply the technique of [1] or [6] to find sets $E_1^n(\xi), \dots, E_{m_n}^n(\xi)$ such that

$$\text{measure}\,[E_j^n(\xi)] = T \cdot \pi_j^n(\xi) \, , \quad \int_0^{l\Delta^n} \chi_{E_j^n(\xi)} \rho^n(\xi_j^n) \, d\tau \leq \pi_j^n(\xi) \int_0^{l\Delta^n} \rho^n(\xi_j^n) \, d\tau + 2^{-n} \varepsilon / m_n \, ,$$

$$\int \chi_{E_j^n(\xi)} \beta_j^n \, d\tau \leq \pi_j^n(\xi) \int \beta_j^n(\xi) \, d\tau + 2^{-n} \varepsilon / m_n \leq [\pi_j^n(\xi) + 1/m_n] 2^{-n} \varepsilon \, .$$

Define

$$x^n(\xi)(t) := y(\xi)(0) + \int_0^t \sum_{j=1}^{m_n} \chi_{E_j^n(\xi)} v^{n-1}(\xi_j^n)\, d\tau\,,$$

obtaining a continuous map $x^n\colon \Xi \to \mathrm{AC}$ such that, reasoning as for the case $n = 1$ above,

$$\int_0^t |x^n(\xi)' - x^{n-1}(\xi)'|\, d\tau \leq \int_0^t \rho^n(\xi)\, d\tau + 5 \cdot 2^{-n} \varepsilon\,,$$

$$d[x^n(\xi)'(t), F(t, x^n(\xi)(t))] \leq k(t) \left[\int_0^t \rho^n(\xi)\, d\tau + 6 \cdot 2^{-n} \varepsilon \right],$$

and, using the estimate obtained above for $\int_0^t \rho^n(\xi)\, d\tau$, obtain the estimates (i) and (ii), thus completing the proof of the claim.

c) From the claim it is clear that

$$\int_0^t |x^n(\xi)' - x^{n-1}(\xi)'|\, d\tau \leq \int_0^t [m(t) - m(\tau)]^{n-1}[(n-1)!]^{-1} \rho(\xi)(\tau)\, d\tau + 6 \cdot 2^{-n} e^{2m(t)} \varepsilon\,,$$

so that the sequence of continuous maps $x^n\colon \Xi \to \mathrm{AC}$ converges uniformly to a continuous map $x\colon \Xi \to \mathrm{AC}$ verifying, by (ii), $x(\xi)'(t) \in F[t, x(\xi)(t)]$ *a.e.* on I. Moreover we have:

$$\int_0^t |x(\xi)' - y(\xi)'|\, d\tau \leq \int_0^t \exp\left[\int_\tau^t k(r)\, dr \right] \rho(\xi)(\tau)\, d\tau + \delta\,. \qquad \blacklozenge$$

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