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Dual-standard subgroups in nonperiodic locally soluble groups

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Teoria dei gruppi. — *Dual-standard subgroups in non-periodic locally soluble groups.* Nota di Stewart E. Stonehewer e Giovanni Zacher, presentata (*) dal Corrisp. G. Zacher.

Abstract. — Let G be a non-periodic locally solvable group. A characterization is given of the subgroups D of G for which the map $X \to X \cap D$, for all $X \leqslant G$, defines a lattice-endomorphism.

Key words: Group; Lattice; Lattice-endomorphism.

Riassunto. — *Sottogruppi dual-standard nei gruppi non periodici localmente risolubili.* Sia G un gruppo non periodico localmente risolubile. Vengono caratterizzati i sottogruppi D di G per cui la posizione $X \to X \cap D$, per tutti gli $X \leqslant G$, definisce un endomorfismo reticolare.

In a given group G, a subgroup is called *dual standard* ($D \underset{\text{d.s.}}{\leqslant} G$) if, for all subgroups X of G, the map $X \to X \cap D$ is a lattice endomorphism of G. Ivanov [2] has shown that a torsion-free locally soluble group G contains a proper non-trivial dual-standard subgroup if and only if G is locally cyclic and non-trivial. Our aim here is to characterize, more generally, the dual-standard subgroups in non-periodic locally soluble groups. For a description of dual-standard subgroups in locally finite groups we refer to [1] and [5].

We begin by recalling from [5] a routine result which is often useful for reducing arguments to finitely generated situations.

Lemma 1. *Let G be a group, $\mathfrak{L}$ a local system of subgroups of G and $D \leqslant G$. Then $D \underset{\text{d.s.}}{\leqslant} G$ if and only if $D \cap L \underset{\text{d.s.}}{\leqslant} L$ for each L in $\mathfrak{L}$.*

A characterization of dual-standard subgroups in finite groups has been given by Zappa [7]; since the result plays a crucial role in our investigation and also for the convenience of the reader, we recall it here. We denote by $\pi(G)$ the set of primes which divide the order of a finite group G. Then we have

Theorem (Zappa). *Let G be a finite group and $D \leqslant G$. If $D \underset{\text{d.s.}}{\leqslant} G$ then $D \lhd G$ and there are Hall subgroups M, H, L of G with H nilpotent, $G = (M \times L) \rtimes H$, $L \leqslant D \leqslant HL$ and $\pi(HL/D) = \pi(D/L)$. Also the Sylow subgroups of H are cyclic or generalized quaternion (in which case $|H \cap D|$ is twice an odd number). Moreover, if Q is any Sylow subgroup of H, then*

i) $\langle l \rangle^Q = \langle l \rangle^{Q \cap D}$ *for all* $l \in L$; *and* ii) $\mathcal{C}_L(Q) = \mathcal{C}_L(Q \cap D)$.

Conversely if $D \lhd G$ and there exist Hall subgroups M, H, L of G satisfying all the above conditions, then $D \underset{\text{d.s.}}{\leqslant} G$.

(*) Nella seduta del 9 dicembre 1989.

A proof of this theorem may also be found in [6, pp. 75-78]; it will be convenient in what follows to refer to it as Zappa's structure theorem, in particular to (i) and (ii) as Zappa's conditions.

LEMMA 2. *Let G be a finite group with $D \lhd G$ and G/D cyclic. Let $Z_1 \leq D$ with Z_1 contained in the centre $Z(G)$ of G, $\pi(G/D) \subseteq \pi(D/Z_1)$ and, for $q \in \pi(G/D) \cap \pi(Z_1)$ let a Sylow q-subgroup of G be cyclic. Then $D/Z_1 \underset{\text{d.s.}}{\leq} G/Z_1$ implies $D \underset{\text{d.s.}}{\leq} G$.*

PROOF. By Lemma 2 in [5], we may assume that $|G:D| = q^\alpha$ (q prime) and, by induction on $|Z_1|$, Z_1 to be of prime order. From our assumptions it follows that the Zappa structure of G/Z_1 will be $G/Z_1 = (S_qZ_1/Z_1) \times (L/Z_1)$, where S_q is a Sylow q-subgroup of G.

Case 1. $|Z_1| = q$. Then we have $G = S_qL_1$, where S_q is cyclic, $L = L_1 \times Z_1$ and $Z_1 \leq S_q$. Let $T_1 \leq L_1$ such that $[T_1, S_q \cap D] \leq T_1$. Then, with $T = Z_1 \times T_1$, $[T, S_q \cap D] \leq T$ and hence $[T, S_q] \leq T$. Therefore $[T_1, S_q] \leq T_1$ and Zappa's condition (i) holds. Now let $[T_1, S_q \cap D] = 1$ for some $T_1 \leq L_1$. Then $[T_1 \times Z_1, S_q \cap D] = 1$ and since S_q normalizes T_1 by (i), $[T_1 \times Z_1, S_q] \leq Z_1 \cap T_1 = 1$ and (ii) holds.

Case 2. $|Z_1| = p \neq q$. This time $S_q \cap Z_1 = 1$ and $G = S_qL$. Let $T \leq L$ and $[T, S_q \cap \cap D] = 1$. Then $[T, S_q] \leq Z_1$, hence $1 = [T, S_q, S_q] = [T, S_q]$ since q does not divide $|T|$, and (ii) holds. Thus it remains to show that condition (i) holds.

By the Frattini argument it sufficies to show that any r-subgroup T of L (r prime), which is normalized by $S_q \cap D$, is also normalized by S_q. If $r \neq p$, then $[S_q, T \times Z_1] \leq \leq T \times Z_1$ implies $[S_q, T] \leq T$. If $r = p$, then we may assume that $Z_1 \leq T$. Let $T_1 = Z_1 \times T$. Thus T_1 is normalized by S_q. Let

$$U = \bigcap_{x \in S_q} T^x .$$

Then $U \lhd S_qT_1$ and so, factoring by U, we may assume that $U = 1$ and T is elementary abelian. Now $[S_q \cap D, T_1] \leq T^x$ for all $x \in S_q$. Hence $[S_q \cap D, T_1] = 1$ and so $[S_q, T_1] = 1$, by (ii). Thus $[S_q, T] = 1$ and so S_q normalizes T and condition (i) holds.

Therefore in both cases Zappa's theorem gives $D \underset{\text{d.s.}}{\leq} G$ as required. □

We recall that if $D \underset{\text{d.s.}}{\leq} G$, then $D \lhd G$, [6, III Theorem 1] and G/D is periodic if $D \neq 1$ [6, III Theorem 5].

PROPOSITION. *Let G be a non-periodic group and $D (\neq 1)$ be a dual-standard subgroup of G. Suppose that the periodic elements of G form a subgroup $P(G)$. Then $P(G) \leq D$. If in addition G is locally residually finite, then the elements of G/D have orders coprime to those of $P(G)$.*

PROOF. Suppose, for a contradiction, that there is an element g of finite order not contained in D. Let u be an element of infinite order in D and $G_1 = \langle g, u \rangle$. Then $D_1 = G_1 \cap D$ is a dual-standard subgroup of G_1 and $|G_1 : D_1|$ is finite. Let $K = = P(G_1) \cap D_1$. By applying [3, Theorem 3.7] to $D_1/K \underset{\text{d.s.}}{\leq} G_1/K$ we conclude that $g \in D_1$, a contradiction.

Now suppose that G is locally residually finite. To establish the coprimeness property, we may assume that $|G:D| = p$ (prime). Suppose, again for a contradiction, that G has an element b of order p. Then $b \in D$. Also $G = \langle a, D\rangle$ for some element a of infinite order. Thus $G_2 = \langle a, b\rangle$ is non-periodic and $D_2 = G_2 \cap D \underset{\text{d.s.}}{\leqslant} G_2$.

By hypothesis there is a subgroup $X \lhd G_2$ with $|G_2 : X|$ finite and $b \notin X$. Then $P(G_2) \cap X \lhd G_2$ and without loss of generality we may assume that $P(G_2) \cap X = 1$.

Let $1 \neq Z_1 \leqslant \langle a\rangle \cap Z(G_2) \cap D_2$ and $Z_2 = Z_1^p$. Then $D_2/Z_2 \underset{\text{d.s.}}{\leqslant} G_2/Z_2$ and, by Zappa's structure theorem, the Sylow p-subgroup of G_2/Z_2 is cyclic. But $Z_1 P(G_2)/Z_2$ contains a non-cyclic elementary abelian p-subgroup, a contradiction. □

Now we can prove our main result.

THEOREM. *Let G be a non-periodic locally soluble group and D be a non-trivial proper subgroup of G. Then D is a dual standard subgroup of G if and only if the following conditions are satisfied:*

(*a*) *$D \lhd G$ and G/D is periodic;*

(*b*) *the periodic elements of G form a subgroup B contained in D and G/B is locally cyclic;*

(*c*) *the elements of G/D have orders coprime to those of B;*

(*d*) *for all $b \in B$ and $g \in G$*

(i) $\langle b\rangle^{\langle g\rangle} = \langle b\rangle^{\langle g\rangle \cap D}$ *and* (ii) $[b, \langle g\rangle] = 1$ *if* $[b, \langle g\rangle \cap D] = 1$.

PROOF. Suppose first that $D \underset{\text{d.s.}}{\leqslant} G$. As we have already pointed out, (*a*) holds for arbitrary groups G. In order to prove (*b*)-(*d*) we may assume that G is finitely generated, hence soluble. Then (*b*) follows from [3, Theorem 3.7]. Also (*d*) is a special case of [4, Theorem 5]. It remains to prove (*c*). By [4, Theorem 4], B is finite, so G is residually finite. Thus (*c*) follows from the Proposition above.

Conversely, suppose that (*a*)-(*d*) hold. By Lemma 1, we may assume that G is finitely generated and therefore soluble. Thus $G = \langle g, B\rangle$ for some element g of infinite order. Let $|G:D| = m$. Then, by (*d*) (i), for each $b \in B$, $\langle b\rangle^{\langle g\rangle} = \langle b\rangle^{\langle g^m\rangle}$. Therefore, as in the proof of [4, Theorem 4], we see that B is finite. Thus $D \cap Z(G) \cap \langle g\rangle \neq 1$. Let $1 \neq z \in D \cap Z(G) \cap \langle g\rangle$. By choosing z appropriately we may assume that $\pi(G/D) \subseteq \pi(D/\langle z\rangle)$. Then using (*d*) (i) and (ii), it follows from [4, Theorem 7], that $D/\langle z\rangle \underset{\text{d.s.}}{\leqslant} G/\langle z\rangle$.

Note that, by Lemma 2, we may replace $\langle z\rangle$ by any smaller subgroup $\neq 1$. To show that $D \underset{\text{d.s.}}{\leqslant} G$, let $U, V \leqslant G$. If U and V have finite index in G, then we can choose $z \in U \cap V$. Hence $\langle U, V\rangle \cap D = \langle U \cap D, V \cap D\rangle$.

It remains to consider the case where U (say) is infinite and V is finite. Thus $V \leqslant B$ and $U = \langle g^n b\rangle (U \cap B)$ for some $b \in B$ and $n \geqslant 1$. We claim that $\langle U, V\rangle \cap D = \langle U \cap D, V\rangle$. For $\langle U, V\rangle = \langle g^n b, U \cap B, V\rangle$, $U \cap D = (\langle g^n b\rangle \cap D)(U \cap B)$; and by (*d*) (i) we have $\langle U \cap B, V\rangle^{\langle g^n b\rangle} = \langle U \cap B, V\rangle^{\langle g^n b\rangle \cap D}$. But then $\langle U, V\rangle \cap D = (\langle U \cap B, V\rangle^{\langle g^n b\rangle} \langle g^n b\rangle) \cap D = \langle U \cap B, V\rangle^{\langle g^n b\rangle \cap D}(\langle g^n b\rangle \cap D) = \langle U \cap B, V, \langle g^n b\rangle \cap D\rangle = \langle U \cap D, V\rangle$ as claimed. Therefore $D \underset{\text{d.s.}}{\leqslant} G$. □

REMARK. As a consequence of the main result in [5], we show that the group $[G, B]$ is actually a direct product of its Sylow subgroups. In fact, assuming w.l.o.g., that G is finitely generated, we have $G = \langle gb, B \rangle$ with $b \in B$ and B a finite group, whose order is relatively prime to $[G:D]$. Let $T = \mathcal{C}(B) \cap \langle gb \rangle \cap D$; T is normal in G and G/T is a finite group with D/T non-trivial dual-standard subgroup of G/T. By Zappa's structure theorem, we have $T \leqslant BT \leqslant L \leqslant D$ and since by [5], $[gb, L]\, T/T$ must be nilpotent, such is $[gb, B]$, hence also $[G, B]$. □

This paper is dedicated to the memory of Gaetano Scorza on the 50th anniversary of his death.

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