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A propagation theorem for a class of microfunctions

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Equazioni a derivate parziali. — *A propagation theorem for a class of microfunctions.* Nota di ANDREA D'AGNOLO e GIUSEPPE ZAMPIERI, presentata (*) dal Socio G. SCORZA DRAGONI.

ABSTRACT. — Let A be a closed set of $M \simeq \mathbb{R}^n$, whose conormal cones $x + \gamma_x^*(A)$, $x \in A$, have locally empty intersection. We first show in § 1 that $\operatorname{dist}(x, A)$, $x \in M \setminus A$ is a C^1 function. We then represent the microfunctions of $\mathcal{C}_{A|X}$, $X \simeq \mathbf{C}^n$, using cohomology groups of $\mathcal{O}_X$ of degree 1. By the results of § 1-3, we are able to prove in § 4 that the sections of $\mathcal{C}_{A|X}|_{\dot{\pi}^{-1}(x_0)}$, $x_0 \in \partial A$, satisfy the principle of the analytic continuation in the complex integral manifolds of $\{H(\phi_i^{\mathbf{C}})\}_{i=1,\dots,m}$, $\{\phi_i\}$ being a base for the linear hull of $\gamma_{x_0}^*(A)$ in $T_{x_0}^* M$; in particular we get $\Gamma_{A \times_M T_M^* X}(\mathcal{C}_{A|X})|_{\partial A \times_M \dot{T}_M^* X} = 0$. When A is a half space with C^ω-boundary, all of the above results were already proved by Kataoka. Finally for a $\mathcal{E}_X$-module $\mathcal{M}$ we show that $\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{A|X})_p = 0$, $p \in \dot{\pi}^{-1}(x_0)$, when at least one conormal $\theta \in \dot{\gamma}_{x_0}^*(A)$ is non-characteristic for $\mathcal{M}$.

KEY WORDS: Partial differential equations on manifolds; Boundary value problems; Theory of functions.

RIASSUNTO. — *Un teorema di propagazione per una classe di microfunzioni.* Sia A un insieme chiuso di $M \simeq \mathbb{R}^n$, i cui coni conormali $x + \gamma_x^*(A)$, $x \in A$, hanno localmente intersezione vuota. Si prova nel § 1 che $\operatorname{dist}(x, A)$, $x \in M \setminus A$ è una funzione C^1. Si rappresentano poi le microfunzioni di $\mathcal{C}_{A|X}$ ($X \simeq \mathbf{C}^n$), mediante gruppi di coomologia di $\mathcal{O}_X$ in grado 1. Se ne deduce nel § 4 un principio di prolungamento analitico per sezioni di $\mathcal{C}_{A|X}|_{\dot{\pi}^{-1}(x_0)}$, $x_0 \in \partial A$ che generalizza alcuni risultati di Kataoka. Se ne dà infine applicazione ai problemi ai limiti.

§ 1 (1). Let X be a C^∞ manifold, A a closed set of X. We denote by $\gamma(A) \subset TX$ the set

$$\gamma_x(A) = C(A, \{x\}), \qquad x \in X,$$

where $C(A, \{x\})$ is the normal cone to A along $\{x\}$ in the sense of [4]; we also denote by $\gamma^*(A)$ the polar cone to $\gamma(A)$. We assume that in some coordinates in a neighborhood of a point $x_0 \in \partial A$:

(1.1) (i) $(x - \gamma_x^*(A)) \cap (y - \gamma_y^*(A)) \cap S = \emptyset \quad \forall x \neq y \in \partial A \cap S$,

(ii) $x \mapsto \gamma_x^*(A)$ is upper semicontinuous.

REMARK 1.1.

(*a*) If A is convex in $X \cong \mathbb{R}^n$ then (1.1) holds. Moreover in this case $\gamma(A) = \overline{N(A)}$ where $N(A)$ is the normal cone to A in the linear hull of A.

(*b*) All sets A with C^2-boundary satisfy (1.1).

(*c*) The set $A = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \leq -\sqrt{|x_2|}\}$ verifies (1.1). (Here $\gamma_0^*(A) = \mathbb{R}_{x_2}^-$ but $N_0^*(A) = \mathbb{R}^2$.)

(*) Nella seduta del 18 novembre 1989.
(1) This section has been modified on the proofs.

(*d*) The set $A=\{(x_1,x_2)\in\mathbb{R}^2: x_1\le |x_2|^{3/2}\}$ does not verify (1.1); in particular (1.1) is not C^1-coordinate invariant.

LEMMA 1.2. *Fix coordinates in* X *at* x_0 *and assume that* (1.1) *holds. Then for every* $x\in(X\setminus A)\cap S_\varepsilon$ $(S_\varepsilon=\{y: |y-x_0|<\varepsilon\}$, ε *small*) *there exists a unique point* $a=a(x)\in\partial A\cap S_{2\varepsilon}$ *such that*

$$x\in a-\gamma_a^*(A). \tag{1.2}$$

PROOF. One takes a point $a=a(x)$ verifying

$$|x-a|=\operatorname{dist}(x,\partial A), \tag{1.3}$$

and verifies easily that a also verifies (1.2). The unicity is assured by (1.1). ■

From the uniqueness it easily follows that $a(x)$ is a continuous function. (One could even easily prove that it is Lipschitz-continuous.)

We set $d(x)=\operatorname{dist}(x,A)$ and, for $t\ge 0$, $A_t=\{x: d(x)\le t\}$; we also set $\gamma_x=\gamma_x(A_{d(x)})$.

LEMMA 1.3. *Let* (1.1) *hold in some coordinate system; then* γ_x, $x\in\partial A$ *are half-spaces and the mapping* $x\mapsto\gamma_x$ *is continuous.*

PROOF. We shall show that

$$\gamma_x=\{y: \langle y-x, a(x)-x\rangle\ge 0\}. \tag{1.4}$$

(The function $x\mapsto a(x)$ being continuous, the lemma will follow at once.) In fact since $\{y: |y-a(x)|\le d(x)\}\subset A_{d(x)}$, then "$\supseteq$" holds in (1.4). On the other hand we reason by absurd and find a sequence $\{x_\nu\}$ such that

$$\begin{cases} x_\nu\to x, \\ d(x_\nu)\le d(x), \\ \langle a(x)-x, x_\nu-x\rangle\le -\delta|x_\nu-x|, \qquad \delta>0. \end{cases} \tag{1.5}$$

By continuity we can replace $a(x)-x$ by $a(x_\nu)-x_\nu$ in (1.5) and conclude that, for large ν, $|x-a(x_\nu)|<|x_\nu-a(x_\nu)|=d(x_\nu)$, a contradiction. ■

Let $N(A)$ be the normal cone to A in the sense of [4].

LEMMA 1.4. *Let* B *be closed and assume that:*

(1.6) $\gamma_x(B)$ *is a half space for every* $x\in\partial B$,

(1.7) $x\mapsto\gamma_x(B)$ *is continuous.*

Then $N_x(B)$, $x\in\partial B$ *are also half spaces.*

PROOF. Suppose by absurd that there exist $\Gamma'\subset\subset\gamma_{x_0}(B)$ and two sequences $\{z_\nu\}$,

$\{y_\nu\}$ with

$$\begin{cases} z_\nu, y_\nu \to x_0, \\ z_\nu, y_\nu \in \partial B, \\ \theta_\nu = y_\nu - z_\nu \in \operatorname{int}\Gamma', \\ [z_\nu, y_\nu] \subset B. \end{cases}$$

(Here $[z_\nu, y_\nu]$ denotes the segment from z_ν to y_ν.) But then $\gamma_{y_\nu}(B) \supset \Gamma' \cup \{-\theta_\nu\}$, a contradiction. ■

REMARK 1.5. Let B verify $N_{x_0}(B) \neq \{0\}$, then if one takes coordinates with $(0, \dots, 0, 1) \in N_{x_0}(B)$ and sets $x = (x', x_n)$, one can represent $\partial B = \{x : x_n - \varphi(x') = 0\}$ for a Lipschitz-continuous function φ. Moreover if $N_{x_0}(B)$ is a half-space and if we let $\boldsymbol{R}^+(0, \dots, 0, 1) = N^*_{x_0}(B)$, then φ is differentiable at x_0 due to $|\varphi(x') - \varphi(x'_0)| = o(|x' - x'_0|)$.

PROPOSITION 1.6. *Let* (1.1) *hold in some coordinates; then* $d(x)$, $x \notin A$, *is a* C^1 *function.*

PROOF. By Lemmas 1.3, 1.4, $N_x(A_{d(x)})$ are half-spaces; set $\tau_x = \partial N_x(A_{d(x)})$ and denote by n_x the normal to τ_x. Let $y \in \tau_x$; according to Remark 1.5 there exists $\tilde{y} \in \partial A_{d(x)}$ with $|\tilde{y} - y| = o(|y - x|)$. It follows:

$$|d(y) - d(x)| = |d(y) - d(\tilde{y})| \le k|y - \tilde{y}| = o(|y - x|)\,. \tag{1.8}$$

By (1.8) we obtain $(\partial/\partial\tau_x)\, d(x) = 0$. On the other hand one has $(\partial/\partial n_x)\, d(x) = 1$. Finally $\partial d(x) = n_x$, $x \notin A$, and hence d is C^1. ■

§**2.** Let X be a C^∞-manifold, $Y \subset X$ a C^1-submanifold, $M^\cdot$ a complex of $\boldsymbol{Z}$-modules of finite rank, and set $M^{\cdot *} = \boldsymbol{R}\mathcal{H}om_{\boldsymbol{Z}}(M^\cdot, \boldsymbol{Z})$. Let $\mu\text{hom}(\cdot,\cdot)$ be the bifunctor of [4, §5]; one easily proves that

$$\mu\text{hom}(\boldsymbol{Z}_Y, M^\cdot_Y) \cong M^\cdot_{T^*_YX}, \tag{2.1}$$

$$\mu\text{hom}(M^\cdot_Y, \boldsymbol{Z}_Y) \cong M^{\cdot *}_{T^*_YX}. \tag{2.2}$$

LEMMA 2.1. *Let* $M^\cdot_Y \cong \boldsymbol{Z}_Y$ *in* $D^+(X; p)$, $p \in \dot{T}^*_YX$ ([4, §6]). *Then* $M^\cdot \cong \boldsymbol{Z}$.

PROOF. The proof is a straightforward consequence of the formula $Hom_{D^+(X;p)}(\cdot,\cdot) \xrightarrow{\sim} H^0\mu\text{hom}(\cdot,\cdot)_p$, and of (2.1), (2.2). ■

§**3.** Let M be a C^ω-manifold of dimension n, X a complexification of M, $A \subset M$ a closed set. According to [6] we define $\mathcal{C}_{A|X} = \mu\text{hom}(\boldsymbol{Z}_A, \mathcal{O}_X) \otimes \text{or}_{M|X}[n]$.

We assume here that

(3.1) (i) A satisfies (1.1) in some coordinates at $x_0 = 0$,

(ii) $A = \overline{\operatorname{int} A}$ in the linear hull of A,

(iii) $SS(\boldsymbol{Z}_A) \subset \gamma^*(A)$.

We take coordinates $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \cong M$, $(z', z_n) \in \mathbb{C}^n \cong X$, and suppose that $A = A' \times \mathbb{R}$. We define

$$G_A = \{z : y_n \geq \inf_{a' \in A'} a'^2/4 + \langle y', a' \rangle /2\}. \tag{3.2}$$

LEMMA 3.1. *∂G_A is C^1.*

PROOF. One defines the set

$$\{z : y_n \geq -y'^2/4 \text{ for } y' \in -A', y_n \geq a^2(-y')/4 + \langle y', a(-y')\rangle /2 \text{ for } y' \notin -A'\}, \tag{3.3}$$

(where $a(-y')$ is the point of $\partial A'$ such that $-y' \in a(-y') - \gamma^*_{a(-y')}(A')$).

One easily proves that the above set coincides with G_A. Then one observes that the boundary of (3.3) is defined by

$$\begin{cases} -y'/4, & \text{for } y' \in -A' \\ a^2(-y')/4 + \langle y', a(-y')\rangle/2 = |y' + a(-y')|^2/4 - y'^2/4 = \\ \qquad = \operatorname{dist}^2(y', -A')/4 - y'^2/4, & \text{for } y' \notin -A'. \end{cases} \tag{3.4}$$

Since $\operatorname{dist}(y', -A')$, $y' \in M' \setminus -A'$, is C^1, due to Proposition 1.4, then the function defined in (3.4) is also C^1.

PROPOSITION 3.2 (*cf.* [6]):

(i) *We can find a complex homogeneous symplectic transformation ϕ such that*

$$\phi(A \times_M T^*_M X \oplus \gamma^*(A)) = N^*(G_A). \tag{3.5}$$

(ii) *ϕ can be quantized to Φ such that*

$$\Phi(\mathbf{Z}_A) = \mathbf{Z}_{G_A}[n-1]; \tag{3.6}$$

in particular

$$(\mathcal{C}_{A|X})_p \cong \mathcal{H}^1_{G_A}(\mathcal{O}_X)_{\pi(\phi(p))}, \qquad p \in \dot{\pi}^{-1}(x_0). \tag{3.7}$$

PROOF. One takes coordinates $(z, \zeta) \in T^*X$, and defines ϕ for $\operatorname{Im}\zeta_n > 0$ by:

$$\begin{cases} z' \mapsto \zeta'/\zeta_n - \sqrt{-1}\, z', \\ z_n \mapsto \langle z, \zeta/\zeta_n\rangle/2 - \sqrt{-1}\, z'^2/4, \\ \zeta' \mapsto -z'\zeta_n/2, \\ \zeta_n \mapsto \zeta_n. \end{cases}$$

Then by recalling that G_A coincides with the set (3.3), one gets (i). As for (ii) one sets

$\mathcal{F}=\Phi(\mathbf{Z}_A)$, $Y=\partial G_A$, and denotes by $j: Y \hookrightarrow X$ the embedding. One gets $SS(\mathcal{F}) \subset \pi^{-1}(Y)$ at p; hence $\mathcal{F} \cong j_* \mathcal{G}$ at p for some $\mathcal{G}$ in $D^+(Y)$ (cf. [4, §6]). On the other hand one has $SS(\mathcal{G}) \subset T^*_Y Y$ at $\pi(p)$ ([4, Prop. 4.1.1]); hence $\mathcal{G} \cong M^\cdot_Y$ at $\pi(p)$ for a complex $M^\cdot$ of $\mathbf{Z}$-modules.

Due to (3.1) (ii) there exists $q \sim p$ such that A is a manifold at $\pi(q)$ and hence by [4, § 11] we get $\mathcal{G} \cong \mathbf{Z}_Y[n-1]$ at $\phi(q)$. Thus (3.6) follows from Lemma 2.1, and (3.7) from the fact that $X \setminus G_A$ is pseudoconvex.

For convex A, the above proposition is stated in [6].

§4. Let M be a C^ω-manifold, X a complexification of M, $A \subset M$ a closed set satisfying condition (3.1).

PROPOSITION 4.1. *Let $\{\phi_i\}_{i=1,\dots,m}$ be a base for the space spanned by $\gamma^*_{x_0}(A)$ in $T^*_{x_0}X$. Then the sections of $\mathcal{C}_{A|X}|_{(T^*_M X \oplus \gamma^*(A))_{x_0}}$* satisfy the principle of the analytic continuation on the complex integral manifolds of $\{H(\phi_i^C)\}_{i=1,\dots,m}$.

PROOF. Using the trick of the dummy variable we can assume A being of the form $A' \times \mathbf{R}$ and hence use the transformation ϕ of **§3**.

Let $p, q \in \phi(T^*_M X \oplus \gamma^*(A))_{x_0}$ belong to the same integral leaf of $\{H(\phi_i^C)\}_{i=1,\dots,m}$. We then have to show that if f is holomorphic in $X \setminus G_A$ and extends holomorphically at $\pi(p)$, then it also extends at $\pi(q)$.

We observe that $\phi(T^*_M X \oplus \gamma^*(A)) = T^*_{\partial G_A} X$ in $\phi(\dot{\pi}^{-1}(x_0))$; thus the claim follows from the Bochner's tube theorem at least when $\rho_M(p)$, $\rho_M(q)$ belong to the interior of $\gamma^*_{x_0}(A)$ in the plane of $\{\phi_i\}$ ($\rho_M: T^*X \to T^*_M X$).

Otherwise one has to remember that ∂G_A is C^1, and use the following result whose proof is straightforward.

LEMMA 4.2 (*cf.* [1]). *Let $(z_1, z_2) \in \mathbf{C}^2$, $z_i = x_i + \sqrt{-1}\, y_i$, $i=1,2$ and let ψ be a C^1 function on $\mathbf{R}^2_{y_1,y_2}$ at 0 such that $\psi \geq 0$ and $\psi = 0$ for $y_1 \geq 0$. If f is analytic in the set*

$$\{|x_i| < \varepsilon\} \times (\{|y_i| < \varepsilon, y_2 > \psi(y_1)\} \cup \{y_1 = \varepsilon, -\delta < y_2 \leq 0\}),$$

then f is analytic at 0.

REMARK 4.3.

(*a*) When A is a half-space with C^ω-boundary, Proposition 4.1 was already stated in [2].

(*b*) In the situation of Proposition 4.1, one has (cf. [2]):

$$\Gamma_{A \times_M T^*_M X}(\mathcal{C}_{A|X})|_{\partial A \times_M T^*_M X} = 0.$$

(*c*) Let $\mathcal{M}$ be an $\mathcal{E}_X$-module at $p \in \dot{\pi}^{-1}(x_0)$. Suppose that there exists $\theta \in \dot{\gamma}^*_{x_0}(A)$ non-characteristic for $\mathcal{M}$. Then: $\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{A|X})_p = 0$.

(This was announced by Uchida when A is convex and all $\theta \in \dot{N}^*_{x_0}(A)$ (or $\partial \dot{N}^*_{x_0}(A)$) are non-characteristic).

Let now Ω be an open set of M and assume that $A = M \setminus \Omega$ satisfies the hypotheses (3.1). By the distinguished triangle $\mathcal{C}_{A|X} \to \mathcal{C}_{M|X} \to \mathcal{C}_{\Omega|X} \xrightarrow{+1}$, by (3.7), and by the corresponding formula for $\mathcal{C}_{M|X}$, one can state (cf. [6]):

PROPOSITION 4.4: We have $H^0(\mathcal{C}_{\Omega|X}) = (\mathcal{C}_{\Omega|X})_{T^*_M X}$.

By (4.1) and by Remark 4.3 (*c*), one also gets, for a $\mathcal{D}_X$-module $\mathcal{M}$:

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_{\bar{\Omega}})_{x_0} = \{ f \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_\Omega(\mathcal{B}_M))_{x_0} ; SS^{\mathcal{M},0}_{\Omega}(f) \cap \dot{\pi}^{-1}(x_0) = \emptyset \} ,$$

$SS^{\mathcal{M},0}_{\Omega}(f)$ being the microsupport in the sense of [6]. (One needs to assume here Z_Ω cohomologically constructible; but this follows probably from (1.1).)

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