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Global analytic and Gevrey surjectivity of the Mizohata operator $D_2 + ix_2^{2k} D_1$

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Equazioni a derivate parziali. — *Global analytic and Gevrey surjectivity of the Mizohata operator* $D_2 + ix_2^{2k} D_1$. Nota di LAMBERTO CATTABRIGA e LUISA ZANGHIRATI, presentata (*) dal Corrisp. L. CATTABRIGA.

ABSTRACT. — The surjectivity of the operator $D_2 + ix_2^{2k} D_1$ from the Gevrey space $\mathcal{E}^{\{s\}}(\mathbb{R}^2)$, $s \geq 1$, onto itself and its non-surjectivity from $\mathcal{E}^{\{s\}}(\mathbb{R}^3)$ to $\mathcal{E}^{\{s\}}(\mathbb{R}^3)$ is proved.

KEY WORDS: Linear partial differential equations; Surjectivity; Gevrey spaces.

RIASSUNTO. — *Suriettività globale, analitica e di Gevrey, dell'operatore di Mizohata* $D_2 + ix_2^{2k} D_1$. Si prova che l'operatore $D_2 + ix_2^{2k} D_1$ è suriettivo dallo spazio di Gevrey $\mathcal{E}^{\{s\}}(\mathbb{R}^2)$, $s \geq 1$, su sé stesso e che ciò non accade per lo stesso operatore da $\mathcal{E}^{\{s\}}(\mathbb{R}^3)$ ad $\mathcal{E}^{\{s\}}(\mathbb{R}^3)$.

Analytic surjectivity of all linear partial differential operators with constant coefficients on $\mathbb{R}^2$ has been proved by E. De Giorgi and L. Cattabriga [5], where a counterexample for the case of operators in $\mathbb{R}^3$ is also indicated (see also [4] and for other results for operators on open sets of $\mathbb{R}^n$: [10] and [8]). Global Gevrey surjectivity of all linear partial differential operators with constant coefficients on $\mathbb{R}^2$ was subsequentely proved by L. Cattabriga [3] in the case of Gevrey spaces with rational index (see also [2], [15], [1] for the case of operators on $\mathbb{R}^n$). As for the case of linear operators with variable coefficients, results on global Gevrey surjectivity are given by L. Ehrenpreis [6], and in the analytic case by T. Kawai [11].

Here we consider, as a simple example of an operator on $\mathbb{R}^2$ with variable coefficients, the Mizohata operator $D_2 + ix_2^{2k} D_1$, where k is a positive integer and $D_j = -i\partial_j$, $j = 1, 2$, and prove the following

THEOREM. *Let* $\mathcal{E}^{\{s\}}(\mathbb{R}^n)$, $s \geq 1$, *be the space of all* C^∞ *functions* f *on* $\mathbb{R}^n$ *such that for every compact subset* K *of* $\mathbb{R}^n$ *there exists a constant* $A > 0$ *such that*

$$\sup_{x \in K} \sup_{\alpha \in \mathbb{Z}_+^n} A^{-|\alpha|} \alpha!^{-s} |\partial_x^\alpha f(x)| < +\infty .$$

Then for every $s \geq 1$

i) $$(D_2 + ix_2^{2k} D_1)\, \mathcal{E}^{\{s\}}(\mathbb{R}^2) = \mathcal{E}^{\{s\}}(\mathbb{R}^2),$$

ii) $$(D_2 + ix_2^{2k} D_1)\, \mathcal{E}^{\{s\}}(\mathbb{R}^3) \subsetneq \mathcal{E}^{\{s\}}(\mathbb{R}^3).$$

PROOF. Let $f \in \mathcal{E}^{\{s\}}(\mathbb{R}^2)$, $s \geq 1$, and let $\sigma > \max\{s, 1\}$. From a result by H. Komatsu [12], it follows that there exists $v_1 \in \mathcal{E}^{\{\sigma\}}(\mathbb{R}^2)$ such that

$$(1) \qquad \begin{cases} v_1(x_1, 0) = 0, & x_1 \in \mathbb{R}, \\ \partial_x^\gamma [(D_2 + ix_2^{2k} D_1) v_1 - f](x_1, 0) = 0, & \forall \gamma \in \mathbb{Z}_+^2,\ x_1 \in \mathbb{R}. \end{cases}$$

(*) Nella seduta del 13 maggio 1989.

Let

$$h(x_1,x_2)=f(x_1,x_2)-(D_2+ix_2^{2k}D_1)\,v_1(x_1,x_2)\,, \tag{2}$$

and with the new variables in $\mathbb{R}^2$

$$\begin{cases} y_1=x_1, \\ y_2=x_2^{2k+1}/(2k+1)\,, \end{cases}$$

$$g(y_1,y_2)=\begin{cases} h(y_1,[(2k+1)\,y_2]^{1/(2k+1)})\,[(2k+1)\,y_2]^{-2k/(2k+1)} & \text{for } y_2\neq 0, \\ 0 & \text{for } y_2=0. \end{cases}$$

In view of (1), $g\in C^\infty(\mathbb{R}^2)$. Hence, by the global $C^\infty(\mathbb{R}^2)$-surjectivity of all linear partial differential operators with constant coefficients, there exists $w\in C^\infty(\mathbb{R}^2)$ such that $(D_2+iD_1)\,w(y_1,y_2)=g(y_1,y_2)$ on $\mathbb{R}^2$.

Letting $v_2(x_1,x_2)=w(x_1,x_2^{2k+1}/(2k+1))$, $(x_1,x_2)\in\mathbb{R}^2$ it follows that $v_2\in C^\infty(\mathbb{R}^2)$ and

$$(D_2+ix_2^{2k}D_1)\,v_2(x_1,x_2)=x_2^{2k}g(x_2,x_2^{2k+1}/(2k+1))=h(x_1,x_2)\,,\quad (x_1,x_2)\in\mathbb{R}^2. \tag{3}$$

Thus from (2) and (3) it follows that $(D_2+ix_2^{2k}D_1)\,(v_1+v_2)=f$ on $\mathbb{R}^2$, where $v=v_1+v_2\in C^\infty(\mathbb{R}^2)$.

Recalling now that the Mizohata operator $D_2+ix_2^{2k}D_1$ is (analytic and) s-Gevrey hypoelliptic on $\mathbb{R}^2$ for every $s\geqslant 1$ (see for example [13], [14], and also [9]), we conclude that v is in fact in the same space $\mathcal{E}^{\{s\}}(\mathbb{R}^2)$ as f is. This proves part i) of the theorem.

To prove ii), let $f\in\mathcal{E}^{\{s\}}(\mathbb{R}^3)$, $s\geqslant 1$, and define

$$g(x_1,x_2,x_3)=f(x_1,x_2^{2k+1}/(2k+1),x_3)\,x_2^{2k},\qquad (x_1,x_2,x_3)\in\mathbb{R}^3.$$

Since $g\in\mathcal{E}^{\{s\}}(\mathbb{R}^3)$, if $(D_2+ix_2^{2k}D_1)\,\mathcal{E}^{\{s\}}(\mathbb{R}^3)=\mathcal{E}^{\{s\}}(\mathbb{R}^3)$, it would exist $w\in\mathcal{E}^{\{s\}}(\mathbb{R}^3)$ such that $(D_2+ix_2^{2k}D_1)\,w(x_1,x_2,x_3)=g(x_1,x_2,x_3)$ on $\mathbb{R}^3$. Hence $v(x_1,x_2,x_3)=w(x_1,[(2k+1)\,x_2]^{1/(2k+1)},x_3)$ would be a solution in $\mathcal{D}'(\mathbb{R}^3)$ of $(D_2+iD_1)\,v=f$. Since the operator D_2+iD_1 is partially hypoelliptic in $\mathbb{R}^3$ with respect to the variables (x_1,x_2), from the corollary of Theorem 4.1.2 of [7], it follows that $v\in\mathcal{E}^{\{s\}}(\mathbb{R}^3)$. So the operator D_2+iD_1 in $\mathbb{R}^3$ would be s-surjective, which is false, as it has been proved in [3] and [4].

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