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Periodic solutions of the Rayleigh equation with damping of definite sign

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Equazioni differenziali ordinarie. — *Periodic solutions of the Rayleigh equation with damping of definite sign.* Nota di Pierpaolo Omari e Gabriele Villari, presentata (*) dal Corrisp. R. Conti.

Abstract. — The existence of a non-trivial periodic solution for the autonomous Rayleigh equation $\ddot{x} + F(\dot{x}) + g(x) = 0$ is proved, assuming conditions which do not imply that $F(x)x$ has a definite sign for $|x|$ large. A similar result is obtained for the periodically forced equation $\ddot{x} + F(\dot{x}) + g(x) = e(t)$.

Key words: Periodic solutions; Nonlinear ordinary differential equations; Limit-cycles.

Riassunto. — *Soluzioni periodiche dell'equazione di Rayleigh con smorzamento di segno definito.* Si dimostra l'esistenza di soluzioni periodiche non costanti per l'equazione di Rayleigh autonoma $\ddot{x} + F(\dot{x}) + g(x) = 0$ senza supporre che $F(x)x$ abbia segno definito per grandi valori di $|x|$. Analogo risultato si ottiene per l'equazione $\ddot{x} + F(\dot{x}) + g(x) = e(t)$ con $e(t)$ periodica.

0. Introduction

In this paper we are concerned with the existence of non-trivial periodic solutions for the autonomous Rayleigh equation

$$\ddot{x} + F(\dot{x}) + g(x) = 0\,, \tag{0.1}$$

where $F, g\colon \mathbb{R} \to \mathbb{R}$ are functions satisfying suitable regularity conditions.

This problem has been investigated by several authors [4, 5, 12, 16, 17, 19], also for its interest in some questions arising in applied sciences [10, 20]. The classical techniques make use of a phase plane analysis for the equivalent system

$$\dot{x} = y\,, \qquad \dot{y} = -F(y) - g(x)\,. \tag{0.2}$$

In this context the usual assumptions on $g(x)$ guarantee that system (0.2) has a unique singular point and its trajectories are clockwise. On the other hand, conditions on $F(x)$ are assumed in order that the singular point is a source and there exists a bounded trajectory. In this situation the Poincaré-Bendixson theorem applies and yields the existence of a stable limit-cycle for system (0.2), which is equivalent to the existence of a non-trivial periodic solution for eq. (0.1). In the literature (see *e.g.* [17]), this goal is usually achieved assuming (essentially)

$$g(x)\,x > 0\,, \qquad \textit{for } x \neq 0\,,$$

$$F(x)\,x < 0\,, \qquad \textit{for } |x| \textit{ small}\,,$$

and

$$F(x)\,x > 0\,, \qquad \textit{for } |x| \textit{ large}\,.$$

Here we prove a theorem where the last condition is not required, being replaced by some assumptions which relate the asymptotic behaviour of $F(x)$ and $g(x)$. It is interesting to stress that, when $g(x)$ is an odd polynomial, $F(x)$ may have a definite sign.

(*) Nella seduta del 16 giugno 1989.

This result is obtained using the so-called property (K) (introduced in [24]), which assumes the existence of a trajectory of system (0.2) coming from infinity and oscillatory for increasing time.

In the final part of the paper, the periodically forced Rayleigh equation

$$\ddot{x} + F(\dot{x}) + g(x) = e(t) \equiv e(t+T) \qquad (T>0) \tag{0.3}$$

is considered. The existence of T-periodic solutions to eq. (0.3) has been studied widely, both using phase plane analysis [1,15,17,18] and functional analytic techniques [3,5,21,7,13,9,8,25,26] (the last three papers deal with systems). Here, the existence of a periodic solution for equation (0.3) is proved using information established by means of property (K) for the related autonomous equation. In this way a flow-invariant region is constructed in order to apply the Brouwer fixed point theorem.

1. The autonomous case

The equation $\ddot{x} + F(\dot{x}) + g(x) = 0$ is equivalent to the phase-plane system

$$\dot{x} = y, \qquad \dot{y} = -F(y) - g(x). \tag{1.1}$$

Throughout, we assume that $F, g\colon \mathbb{R} \to \mathbb{R}$ *are continuous functions satisfying suitable regularity conditions in order to guarantee the uniqueness of the solutions for the Cauchy problems*. Moreover, we suppose that

$$(h_1) \qquad (g(x) + F_0)(x - x_0) > 0, \qquad \textit{for every } x \neq x_0,$$

where $F_0 = F(0)$ and x_0 is such that $g(x_0) = -F_0$. This condition implies that system (1.1) has a unique singular point $S = (x_0, 0)$ and that the trajectories are clockwise.

For every point $P = (x, y) \in \mathbb{R}^2$ denote by $\gamma^+(P)$ and $\gamma^-(P)$ the positive, respectively negative, semitrajectory of system (1.1) passing through P. Then, we give the following definition, which was introduced in [24] for the Liénard system:

System (1.1) *has the property* (K) *if there is a point P such that $\gamma^-(P)$ does not intersect the x-axis and $\gamma^+(P)$ is oscillatory.*

Property (K) may be used for getting the existence of periodic solutions to system (1.1); indeed, if the singular point is a source and property (K) holds, then it is possible to produce at least a limit-cycle by virtue of the Poincaré-Bendixson theorem.

In the following lemma we use a method which was developed by F. Bucci in [2] for the generalized Liénard equation. A first approach in this direction may be found in [22] and [23].

Lemma 1. *Assume (h_1) and consider a point $P = (x_0, y)$, with $y > 0$. Moreover, suppose that*

(h_2) *there is a constant $M > 0$ such that* $\mathrm{F(x)} \geq -\mathrm{M}$, *for* $x > 0$,

and

(h_3) *there exists a continuously differentiable function $\phi(x)$ such that*

$$\phi(x) > 0 \quad \textit{and} \quad F(\phi(x)) > -\phi(x)\,\phi'(x) - g(x), \qquad \textit{for } x \leq x_1 < x_0.$$

Then $\gamma^-(P)$ does not intersect the x-axis, if $y > 0$ is large enough.

PROOF. Consider the points $(x, \phi(x))$, with $x \leqslant x_1$. The slope of a trajectory of system (1.1) at such points is given by $-(F(\phi(x)) + g(x))/\phi(x)$. On the other hand, assumption (h_3) may be written as $-(F(\phi(x)) + g(x))/\phi(x) < \phi'(x)$. This implies that, for every point $Q = (x_1, y)$, with $y > \phi(x_1)$, $\gamma^-(Q)$ does not intersect the graph of $\phi(x)$ and, therefore, the x-axis. Moreover, by (h_2), it is easy to prove that $\gamma^+(Q)$ intersects the line $x = x_0$, at a point (x_0, y_1), with $y_1 \geqslant 0$. Then $\gamma^-(P)$ does not intersect the x-axis, if $y > y_1$. Q.E.D.

REMARK 1.1. It is interesting for the applications to specialize the function $\phi(x)$ in (h_3). If we set $\phi(x) = -x$, we obtain, for $x < 0$, $F(-x) > -g(x) - x$, that is, for $x > 0$, $F(x) > x - g(-x)$.

It we set $\phi(x) = -g(x)$, we have, for $x < 0$,

$$F(-g(x)) > -g(x)[1 + g'(x)], \tag{1.2}$$

provided that, of course, $g(x)$ is continuously differentiable and $g(x) < 0$, for $x < 0$.

It is now interesting to specialize $g(x)$. Setting $g(x) = x^{2p+1}$, with p any positive integer, from (1.2) we get, for $x < 0$,

$$F(-x^{2p+1}) > -x^{2p+1} - (2p+1)\,x^{4p+1}. \tag{1.3}$$

Notice that if

$$F(x) > x^2, \tag{1.4}$$

for $x > 0$ large, then (1.3) is fulfilled. More in general, condition (1.4) implies (h_3), for any polynomial of odd degree, satisfying (h_1). In the special case $g(x) = x$, from (1.2) we derive that

$$F(x) > 2x, \tag{1.5}$$

for $x > 0$ large, implies (h_3). Finally, we remark that if (1.4) or (1.5) hold, then (h_2) is satisfied.

LEMMA 2. *Assume (h_1) and (h_2) and consider a point $P = (x_0, y)$, with $y > 0$. Moreover, suppose that*

(h_4) $\quad (F(x) - F_0)\,x < 0, \quad$ *for* $0 < |x| < \varepsilon$;

(h_5) $\quad \lim_{x \to +\infty} \int_0^x (g(\xi) - M)\, d\xi = +\infty \quad$ *(with M given in (h_2))*;

(h_6) *there exists $k > 0$ such that $(F(x) - F_0) < k|x|$, for $x < 0$, and* $\liminf_{x \to -\infty} g(x)/x > k^2$.

Then $\gamma^+(P)$ intersects the x-axis at $x < x_0$.

PROOF. Consider the auxiliary system

$$\dot{x} = y, \qquad \dot{y} = -(g(x) + M). \tag{1.6}$$

From assumption (h_5) and classical results [11], we know that every trajectory of system (1.6), which passes through a point of the line $x = x_0$ intersects the x-axis at $x > x_0$: a comparison argument gives the same result for system (1.1). Therefore $\gamma^+(P)$ intersects the x-axis at $x > x_0$.

Next, we observe that, by (h_1) and (h_4), the singular point $S = (x_0, 0)$ is a source.

Then, using the former condition in (h_6), we easily get that $\gamma^+(P)$ intersects the line $x = x_0$, at a point $R = (x_0, y_0)$, with $y_0 < 0$.

It remains to prove that $\gamma^+(R) = \gamma^+(P)$ intersects the x-axis at $x < x_0$. To this end, we consider the line $y = kx + h$, where h is such that $kx_0 + h = y_0$. Using again (h_1) and the former condition in (h_6), we derive that, in the half plane $y < 0$, the slope of the trajectory, given by $-(F(y) - F_0 + g(x) + F_0)/y$ is strictly less than k, for $x < x_0$. Hence, $\gamma^+(P)$ does not intersect the line $y = kx + h$, for $x < x_0$. Finally, using both conditions in (h_6), we obtain that, for $x < x_2$, with $x_2 < 0$ large enough, and $y < 0$, the slope of the trajectory is less than $k + (kx + h)^{-1}(k^2 + \delta)|x| \leq -\delta/(2k)$, for some $\delta > 0$. This implies that $\gamma^+(P)$ intersects the x-axis at $x < x_0$. Q.E.D.

THEOREM 1. *Assume that* (h_1)-(h_6) *hold. Then system* (1.1) *has at least a periodic solution which is stable.*

PROOF. It suffices to prove that system (1.1) has property (K). From assumptions (h_1) and (h_4), it follows that there is a unique singular point $S = (x_0, 0)$, which is a source. Then we consider a point $P = (x_0, y)$ with $y > 0$ large enough. By (h_2), (h_3), (h_5) and (h_6), applying Lemmas 1 and 2, we obtain that $\gamma^-(P)$ does not intersect the x-axis at $x < x_0$, while $\gamma^+(P)$ intersects the x-axis at $x < x_0$. Therefore, we have proved that $\gamma^+(P)$ is bounded. Thus the Poincaré-Bendixson theorem implies that system (1.1) has at least a stable limit-cycle. Q.E.D.

REMARK 1.2. Arguing in the same way, it is possible to produce a similar version of Theorem 1, based on a function $\phi(x)$ defined for $x > x_0$ and negative.

REMARK 1.3. With respect to the former condition in (h_6), we point out that it cannot be dropped in general: indeed, if we do not require $(F(x) - F_0) < k|x|$, for $x < 0$, system (1.1) may have a vertical asymptote.

EXAMPLE. Here we present a possible application of Theorem 1. Take numbers α, β, γ, δ, such that $-1 < \alpha < 0$, $|\alpha| < \beta$, $0 < \gamma$ and $\delta = \alpha + \beta - \gamma$, and define:

$$F(x) = \alpha x + \beta, \text{ for } x \leq 1, \qquad F(x) = \gamma x^2 + \delta, \text{ for } x > 1.$$

Set also, for some integer $p \geq 0$, $g(x) = (x-1)^{2p+1} - \beta$.

Clearly, all the assumptions of Theorem 1 are fulfilled; hence system (1.1) has at least a stable periodic solution. As far as we know, no result in this direction, with $F(x)$ positive, for every x, and unbounded, is available in the literature.

In the light of Remark 1.2, examples in which $F(x)$ is negative, for every x, can be produced as well.

2. THE FORCED CASE

Now we consider the problem of the existence of a T-periodic solution $(T > 0)$ to the forced equation

$$\ddot{x} + F(\dot{x}) + g(x) = e(t) \equiv e(t + T), \tag{2.1}$$

where F, g are like in section 1 and $e: \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic.

We follow the classical technique of producing a flow-invariant region in the plane for the equivalent system

$$\dot{x} = y, \qquad \dot{y} = -F(y) - g(x) + e(t). \tag{2.2}$$

The existence of a T-periodic solution for system (2.2) is obtained finding a fixed point for the associated Poincaré map.

THEOREM 2. *Assume*

(k_1) $g(x) > E - F_0$, *for* $x > d (>0)$, *and* $g(x) < e - F_0$, *for* $x < -d$, *where* $F_0 = F(0)$, $E = \max\{e(t): t \in \mathbb{R}\}$ *and* $e = \min\{e(t): t \in \mathbb{R}\}$;

(k_2) *there is a constant* $M > 0$ *such that* $F(x) \geqslant -M$, *for* $x > 0$;

(k_3) *there exists a continuously differentiable function* $\phi(x)$ *such that* $\phi(x) > 0$ *and* $F(\phi(x)) > -\phi(x)\phi'(x) - g(x) + E$, *for* $x \leqslant -d$;

(k_4) $\lim\limits_{x \to +\infty} \int_0^x (g(\xi) - E - M)\, d\xi = +\infty$;

(k_5) *there exists* $k > 0$ *such that* $(F(x) - F_0) < k|x|$, *for* $x < 0$, *and* $\liminf\limits_{x \to -\infty} g(x)/x > k^2$.

Then equation (2.2) *has at least a* T*-periodic solution.*

PROOF. We assume $e < E$, otherwise the result is trivial because from (k_1) we get the existence of a singular point for system (2.2) which, in this case, is autonomous. Then, we introduce the auxiliary systems

$$\dot{x} = y, \qquad \dot{y} = -F(y) - g(x) + E, \tag{2.3}$$

and

$$\dot{x} = y, \qquad \dot{y} = -F(y) - g(x) + e. \tag{2.4}$$

It can be easily seen, by a comparison argument, that for each point $P = (x, y)$, with $y > 0$ (resp. $y < 0$), the slope of the solution of system (2.2), passing through P at any time, does not exceed the slope of the trajectory of system (2.3) (resp. (2.4)) through the same point. Therefore solutions of system (2.2) are guided, in the half-plane $y > 0$, by the corresponding trajectories of system (2.3), while in the half-plane $y < 0$ this happens with trajectories of system (2.4).

Consider system (2.3): arguing like in the previous section, we have that, for every point $P_1 = (-d, y_1)$, with $y_1 > \phi(-d)$, $\gamma^-(P_1)$ does not intersect the x-axis. Then consider the line $y = (m_1/y_1)(x + d) + y_1$, with $m_1 = M + \max\{|g(x) - E| : |x| \leqslant d\}$, and let $P_2 = (d, 2(m_1/y_1)\, d + y_1)$ be its intersection point with the line $x = d$. Consider again system (2.3): like before we obtain that $\gamma^+(P_2)$ intersects the x-axis at some point $P_3 = (x_3, 0)$, with $x_3 > d$. Consider now system (2.4): as before, we have that $\gamma^+(P_3)$ intersects the line $x = d$, at some point $P_4 = (d, y_4)$, with $y_4 < 0$. Then, consider the line $y = (m_2/|y_4|)(x - d) + y_4$, with $m_2 = k|y_4| + \max\{|g(x) - e| : |x| \leqslant d\}$, and let $P_5 =$

$=(-d, -2(m_2/|y_4|)\,d+y_4)$ be its intersection point with the line $x=-d$. Consider again system (2.4): like before we obtain that $\gamma^+(P_5)$ intersects the x-axis at some point $P_6=(x_6,0)$, with $x_6<-d$. Finally, consider system (2.3): clearly, $\gamma^+(P_6)$ intersects the line $x=-d$ at some point $P_7=(-d,y_7)$, with $y_7<\phi(-d)$. Then, we construct the simply connected region V, bounded by the segment P_1P_2, the arc P_2P_3 of trajectory of system (2.3), the arc P_3P_4 of trajectory of system (2.4), the segment P_4P_5, the arc P_5P_6 of trajectory of system (2.4), the arc P_6P_7 of trajectory of system (2.3) and the segment P_7P_1. It is easy to check that V is flow-invariant for system (2.2). Hence the Brouwer fixed point theorem, applied to the associated Poincaré map (T-time map), provides the existence of a T-periodic solution for system (2.2), that is a T-periodic solution for equation (2.1). Q.E.D.

Remark 2.1. Again it is possible to produce a similar version of Theorem 2 based on a function $\phi(x)$ defined for $x>d$ and negative. In this case, we start the construction of the boundary of V from a point $Q_1=(d,y_1)$, with $y_1<0$ large enough.

Remark 2.2. As observed in Remarks 1.2 and 1.3, Theorem 2 can be applied to cases where $F(x)$ is always positive (resp. negative) and unbounded.

Remark 2.3. From the construction of the flow-invariant region V, it follows that Theorem 2 may be invoked to assert that every solution of equation (2.1) is bounded in the future. This result is still true if the forcing term $e(t)$ is not periodic, provided that $e\leqslant e(t)\leqslant E$, for every $t\in\mathbb{R}$.

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