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Periodic solutions of the Rayleigh equation with damping of definite sign


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Abstract. — The existence of a non-trivial periodic solution for the autonomous Rayleigh equation \( \ddot{x} + F(\dot{x}) + g(x) = 0 \) is proved, assuming conditions which do not imply that \( F(x)x \) has a definite sign for \( |x| \) large. A similar result is obtained for the periodically forced equation \( \ddot{x} + F(\dot{x}) + g(x) = e(t) \).

Key words: Periodic solutions; Nonlinear ordinary differential equations; Limit-cycles.

Riassunto. — Soluzioni periodiche dell’equazione di Rayleigh con smorzamento di segno definito. Si dimostra l’esistenza di soluzioni periodiche non costanti per l’equazione di Rayleigh autonoma \( \ddot{x} + F(\dot{x}) + g(x) = 0 \) senza supporre che \( F(x)x \) abbia segno definito per grandi valori di \( |x| \). Analogo risultato si ottiene per l’equazione \( \ddot{x} + F(\dot{x}) + g(x) = e(t) \) con \( e(t) \) periodica.

0. Introduction

In this paper we are concerned with the existence of non-trivial periodic solutions for the autonomous Rayleigh equation

\[
\ddot{x} + F(\dot{x}) + g(x) = 0,
\]

where \( F, g: \mathbb{R} \rightarrow \mathbb{R} \) are functions satisfying suitable regularity conditions.

This problem has been investigated by several authors [4,5,12,16,17,19], also for its interest in some questions arising in applied sciences [10,20]. The classical techniques make use of a phase plane analysis for the equivalent system

\[
\dot{x} = y, \quad \dot{y} = -F(y) - g(x).
\]

In this context the usual assumptions on \( g(x) \) guarantee that system (0.2) has a unique singular point and its trajectories are clockwise. On the other hand, conditions on \( F(x) \) are assumed in order that the singular point is a source and there exists a bounded trajectory. In this situation the Poincaré-Bendixson theorem applies and yields the existence of a stable limit-cycle for system (0.2), which is equivalent to the existence of a non-trivial periodic solution for eq. (0.1). In the literature (see e.g. [17]), this goal is usually achieved assuming (essentially)

\[
g(x)x > 0, \quad F(x)x < 0, \quad F(x)x > 0,
\]

for \( x \neq 0 \), for \( |x| \) small, for \( |x| \) large.

Here we prove a theorem where the last condition is not required, being replaced by some assumptions which relate the asymptotic behaviour of \( F(x) \) and \( g(x) \). It is interesting to stress that, when \( g(x) \) is an odd polynomial, \( F(x) \) may have a definite sign.

This result is obtained using the so-called property (K) (introduced in [24]), which assumes the existence of a trajectory of system (0.2) coming from infinity and oscillatory for increasing time.

In the final part of the paper, the periodically forced Rayleigh equation

\[ (0.3) \quad \dot{x} + F(x) + g(x) = e(t) \equiv e(t + T) \quad (T > 0) \]

is considered. The existence of $T$-periodic solutions to eq. (0.3) has been studied widely, both using phase plane analysis [1, 15, 17, 18] and functional analytic techniques [3, 5, 21, 7, 13, 9, 8, 25, 26] (the last three papers deal with systems). Here, the existence of a periodic solution for equation (0.3) is proved using information established by means of property (K) for the related autonomous equation. In this way a flow-invariant region is constructed in order to apply the Brouwer fixed point theorem.

1. THE AUTONOMOUS CASE

The equation $\dot{x} + F(x) + g(x) = 0$ is equivalent to the phase-plane system

\[ (1.1) \quad \dot{x} = y, \quad \dot{y} = -F(y) - g(x). \]

Throughout, we assume that $F, g: \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying suitable regularity conditions in order to guarantee the uniqueness of the solutions for the Cauchy problems. Moreover, we suppose that

\[ (b_1) \quad (g(x) + F_0)(x - x_0) > 0, \quad \text{for every } x \neq x_0, \]

where $F_0 = F(0)$ and $x_0$ is such that $g(x_0) = -F_0$. This condition implies that system (1.1) has a unique singular point $S = (x_0, 0)$ and that the trajectories are clockwise.

For every point $P = (x, y) \in \mathbb{R}^2$ denote by $\gamma^-(P)$ and $\gamma^+(P)$ the positive, respectively negative, semitrajectory of system (1.1) passing through $P$. Then, we give the following definition, which was introduced in [24] for the Liénard system:

System (1.1) has the property (K) if there is a point $P$ such that $\gamma^-(P)$ does not intersect the x-axis and $\gamma^+(P)$ is oscillatory.

Property (K) may be used for getting the existence of periodic solutions to system (1.1); indeed, if the singular point is a source and property (K) holds, then it is possible to produce at least a limit-cycle by virtue of the Poincaré-Bendixson theorem.

In the following lemma we use a method which was developed by F. Bucci in [2] for the generalized Liénard equation. A first approach in this direction may be found in [22] and [23].

Lemma 1. Assume (b$_1$) and consider a point $P = (x_0, y)$, with $y > 0$. Moreover, suppose that

\[ (b_2) \quad \text{there is a constant } M > 0 \text{ such that } F(x) \geq -M, \quad \text{for } x > 0, \]

and

\[ (b_3) \quad \text{there exists a continuously differentiable function } \dot{\phi}(x) \text{ such that } \]

\[ \dot{\phi}(x) > 0 \quad \text{and} \quad F(\dot{\phi}(x)) > -\dot{\phi}(x) \dot{\phi}'(x) - g(x), \quad \text{for } x \leq x_1 < x_0. \]

Then $\gamma^-(P)$ does not intersect the x-axis, if $y > 0$ is large enough.
Proof. Consider the points \( (x, \phi(x)) \), with \( x \leq x_1 \). The slope of a trajectory of system (1.1) at such points is given by \(- (F(\phi(x)) + g(x))/\phi(x)\). On the other hand, assumption (b₂) may be written as \(- (F(\phi(x)) + g(x))/\phi(x) < \phi'(x)\). This implies that, for every point \( \gamma \geq (x_1, y) \), with \( \gamma > \phi(x_1) \), \( \gamma^- (Q) \) does not intersect the graph of \( \phi(x) \) and, therefore, the x-axis. Moreover, by (b₂), it is easy to prove that \( \gamma^+ (Q) \) intersects the line \( x = x_0 \), at a point \( (x_0, y_1) \), with \( y_1 > 0 \). Then \( \gamma^- (P) \) does not intersect the x-axis, if \( y > y_1 \). Q.E.D.

Remark 1.1. It is interesting for the applications to specialize the function \( \phi(x) \) in (b₃). If we set \( \phi(x) = -x \), we obtain, for \( x < 0 \), \( F(-x) > -g(x) - x \), that is, for \( x > 0 \), \( F(x) > x - g(-x) \).

If we set \( \phi(x) = -g(x) \), we have, for \( x < 0 \),

\[
F(-g(x)) > - g(x) [1 + g'(x)],
\]

provided that, of course, \( g(x) \) is continuously differentiable and \( g(x) < 0 \), for \( x < 0 \).

It is now interesting to specialize \( g(x) \). Setting \( g(x) = x^{2p+1} \), with \( p \) any positive integer, from (1.2) we get, for \( x < 0 \),

\[
F(-x^{2p+1}) > -x^{2p+1} - (2p + 1)x^{4p+1}.
\]

Notice that if

\[
F(x) > x^2,
\]

for \( x > 0 \) large, then (1.3) is fulfilled. More in general, condition (1.4) implies (b₃), for any polynomial of odd degree, satisfying (b₁). In the special case \( g(x) = x \), from (1.2) we derive that

\[
F(x) > 2x,
\]

for \( x > 0 \) large, implies (b₃). Finally, we remark that if (1.4) or (1.5) hold, then (b₂) is satisfied.

Lemma 2. Assume (b₁) and (b₂) and consider a point \( P = (x_0, y) \), with \( y > 0 \). Moreover, suppose that

(b₄) \( (F(x) - F_0)x < 0 \), for \( 0 < |x| < \varepsilon \);

(b₅) \( \lim_{x \to -\infty} \int_0^x (g(\xi) - M) \, d\xi = +\infty \) (with \( M \) given in (b₂));

(b₆) there exists \( k > 0 \) such that \( (F(x) - F_0) < k|x| \), for \( x < 0 \), and \( \lim \inf_{x \to -\infty} g(x)/x > k^2 \).

Then \( \gamma^+ (P) \) intersects the x-axis at \( x < x_0 \).

Proof. Consider the auxiliary system

\[
\dot{x} = y, \quad \dot{y} = -(g(x) + M).
\]

From assumption (b₃) and classical results [11], we know that every trajectory of system (1.6), which passes through a point of the line \( x = x_0 \) intersects the x-axis at \( x > x_0 \): a comparison argument gives the same result for system (1.1). Therefore \( \gamma^+ (P) \) intersects the x-axis at \( x > x_0 \).

Next, we observe that, by (b₁) and (b₄), the singular point \( S = (x_0, 0) \) is a source.
Then, using the former condition in \((h_6)\), we easily get that \(\gamma^+(P)\) intersects the line \(x = x_0\), at a point \(R = (x_0, y_0)\), with \(y_0 < 0\).

It remains to prove that \(\gamma^+(R) = \gamma^+(P)\) intersects the \(x\)-axis at \(x < x_0\). To this end, we consider the line \(y = kx + b\), where \(b\) is such that \(kx_0 + b = y_0\). Using again \((b_1)\) and the former condition in \((b_6)\), we derive that, in the half plane \(y < 0\), the slope of the trajectory, given by \(- (F(y) - F_0 + g(x) + F_0)/y\) is strictly less than \(k\), for \(x < x_0\). Hence, \(\gamma^+(P)\) does not intersect the line \(y = kx + b\), for \(x < x_0\). Finally, using both conditions in \((b_6)\), we obtain that, for \(x < x_2\), with \(x_2 < 0\) large enough, and \(y < 0\), the slope of the trajectory is less than \(k + (kx + b)^{-1}(k^2 + \delta)|x| \leq -\delta/(2k)\), for some \(\delta > 0\). This implies that \(\gamma^+(P)\) intersects the \(x\)-axis at \(x < x_0\).

\[\text{Q.E.D.}\]

**THEOREM 1.** Assume that \((b_1)-(b_6)\) hold. Then system \((1.1)\) has at least a periodic solution which is stable.

**Proof.** It suffices to prove that system \((1.1)\) has property \((K)\). From assumptions \((b_1)\) and \((b_4)\), it follows that there is a unique singular point \(S = (x_0, 0)\), which is a source. Then we consider a point \(P = (x_0, y)\) with \(y > 0\) large enough. By \((b_2), (b_3), (b_5)\) and \((b_6)\), applying Lemmas 1 and 2, we obtain that \(\gamma^-(P)\) does not intersect the \(x\)-axis at \(x < x_0\), while \(\gamma^+(P)\) intersects the \(x\)-axis at \(x < x_0\). Therefore, we have proved that \(\gamma^+(P)\) is bounded. Thus the Poincaré-Bendixson theorem implies that system \((1.1)\) has at least a stable limit-cycle. \(\text{Q.E.D.}\)

**Remark 1.2.** Arguing in the same way, it is possible to produce a similar version of Theorem 1, based on a function \(\phi(x)\) defined for \(x > x_0\) and negative.

**Remark 1.3.** With respect to the former condition in \((b_6)\), we point out that it cannot be dropped in general: indeed, if we do not require \((F(x) - F_0) < k|x|\), for \(x < 0\), system \((1.1)\) may have a vertical asymptote.

**Example.** Here we present a possible application of Theorem 1. Take numbers \(\alpha, \beta, \gamma, \delta\), such that \(-1 < \alpha < 0\), \(|\alpha| < \beta\), \(0 < \gamma\) and \(\delta = \alpha + \beta - \gamma\), and define:

\[F(x) = \alpha x + \beta, \quad \text{for } x \leq 1, \quad F(x) = \gamma x^2 + \delta, \quad \text{for } x > 1.\]

Set also, for some integer \(p \geq 0\), \(g(x) = (x - 1)^{2p+1} - \beta\).

Clearly, all the assumptions of Theorem 1 are fulfilled; hence system \((1.1)\) has at least a stable periodic solution. As far as we know, no result in this direction, with \(F(x)\) positive, for every \(x\), and unbounded, is available in the literature.

In the light of Remark 1.2, examples in which \(F(x)\) is negative, for every \(x\), can be produced as well.

2. **The forced case**

Now we consider the problem of the existence of a \(T\)-periodic solution \((T > 0)\) to the forced equation

\[\ddot{x} + F(\dot{x}) + g(x) = e(t) \equiv e(t + T),\]

where \(F, g\) are like in section 1 and \(e: \mathbb{R} \rightarrow \mathbb{R}\) is continuous and \(T\)-periodic.
We follow the classical technique of producing a flow-invariant region in the plane for the equivalent system
\begin{align}
\dot{x} &= y, \\
\dot{y} &= -F(y) - g(x) + e(t).
\end{align}

The existence of a $T$-periodic solution for system (2.2) is obtained finding a fixed point for the associated Poincaré map.

**Theorem 2.** Assume
\begin{enumerate}
\item[(k_1)] $g(x) > E - F_0$, for $x > d(>0)$, and $g(x) < e - F_0$, for $x < -d$;
\item[(k_2)] there is a constant $M > 0$ such that $F(x) \geq -M$, for $x > 0$;
\item[(k_3)] there exists a continuously differentiable function $\varphi(x)$ such that $\dot{\varphi}(x) > 0$ and $F(\varphi(x)) > -\varphi(x) \varphi'(x) - g(x) + E$, for $x \leq -d$;
\item[(k_4)] \[ \lim_{x \to +\infty} \int_0^x (g(\xi) - E - M) d\xi = +\infty; \]
\item[(k_5)] there exists $k > 0$ such that 
\[ (F(x) - F_0) < k|x|, \text{ for } x < 0, \text{ and } \liminf_{x \to -\infty} g(x)/x > k^2. \]
\end{enumerate}

Then equation (2.2) has at least a $T$-periodic solution.

**Proof.** We assume $e < E$, otherwise the result is trivial because from $(k_4)$ we get the existence of a singular point for system (2.2) which, in this case, is autonomous. Then, we introduce the auxiliary systems
\begin{align}
\dot{x} &= y, \\
\dot{y} &= -F(y) - g(x) + E,
\end{align}

and
\begin{align}
\dot{x} &= y, \\
\dot{y} &= -F(y) - g(x) + e.
\end{align}

It can be easily seen, by a comparison argument, that for each point $P = (x, y)$, with $y > 0$ (resp. $y < 0$), the slope of the solution of system (2.2), passing through $P$ at any time, does not exceed the slope of the trajectory of system (2.3) (resp. (2.4)) through the same point. Therefore solutions of system (2.2) are guided, in the half-plane $y > 0$, by the corresponding trajectories of system (2.3), while in the half-plane $y < 0$ this happens with trajectories of system (2.4).

Consider system (2.3): arguing like in the previous section, we have that, for every point $P_1 = (-d, y_1)$, with $y_1 > \dot{\varphi}(-d)$, $\gamma^-(P_1)$ does not intersect the x-axis. Then consider the line $y = (m_1/y_1)(x + d) + y_1$, with $m_1 = M + \max \{|g(x)| - E|:|x| \leq d\}$, and let $P_2 = (d,2(m_1/y_1)d + y_1)$ be its intersection point with the line $x = d$. Consider again system (2.3): like before we obtain that $\gamma^+(P_2)$ intersects the x-axis at some point $P_3 = (x_3, 0)$, with $x_3 > d$. Consider now system (2.4): as before, we have that $\gamma^+(P_3)$ intersects the line $x = d$, at some point $P_4 = (d, y_4)$, with $y_4 < 0$. Then, consider the line $y = (m_2/y_4)(x - d) + y_4$, with $m_2 = k|y_4| + \max \{|g(x)| - e|:|x| \leq d\}$, and let $P_5 = \ldots$
\[= (-d, -2(m_2/|y_4|) d + y_4) \text{ be its intersection point with the line } x = -d. \]
Consider again system (2.4): like before we obtain that \(\gamma^+(P_5)\) intersects the x-axis at some point \(P_6 = (x_6, 0)\), with \(x_6 < -d\). Finally, consider system (2.3): clearly, \(\gamma^+(P_6)\) intersects the line \(x = -d\) at some point \(P_7 = (-d, y_7)\), with \(y_7 < \phi(-d)\). Then, we construct the simply connected region \(V\), bounded by the segment \(P_1 P_2\), the arc \(P_2 P_3\) of trajectory of system (2.3), the arc \(P_3 P_4\) of trajectory of system (2.4), the segment \(P_4 P_5\), the arc \(P_5 P_6\) of trajectory of system (2.4), the arc \(P_6 P_7\) of trajectory of system (2.3) and the segment \(P_7 P_1\). It is easy to check that \(V\) is flow-invariant for system (2.2). Hence the Brouwer fixed point theorem, applied to the associated Poincaré map (T-time map), provides the existence of a \(T\)-periodic solution for system (2.2), that is a \(T\)-periodic solution for equation (2.1). Q.E.D.

**Remark 2.1.** Again it is possible to produce a similar version of Theorem 2 based on a function \(\phi(x)\) defined for \(x > d\) and negative. In this case, we start the construction of the boundary of \(V\) from a point \(Q_1 = (d, y_1)\), with \(y_1 < 0\) large enough.

**Remark 2.2.** As observed in Remarks 1.2 and 1.3, Theorem 2 can be applied to cases where \(F(x)\) is always positive (resp. negative) and unbounded.

**Remark 2.3.** From the construction of the flow-invariant region \(V\), it follows that Theorem 2 may be invoked to assert that every solution of equation (2.1) is bounded in the future. This result is still true if the forcing term \(e(t)\) is not periodic, provided that \(e \leq e(t) \leq E\), for every \(t \in \mathbb{R}\).

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**References**

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