Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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An existence result for minimal spheres in manifolds boundary

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **1** (1990), n.1, p. 11–15.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1990_9_1_1_11_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1990.

Rend. Mat. Acc. Lincei
s. 9, v. 1:11-15 (1990)

Analisi matematica. — *An existence result for minimal spheres in manifolds with boundary.* Nota (*) di Edi Rosset, presentata dal Socio E. De Giorgi.

Abstract. — We prove the existence of a not homotopically trivial minimal sphere in a 3-manifold with boundary, obtained by deleting an open connected subset from a compact Riemannian oriented 3-manifold with boundary, having trivial second homotopy group.

Key words: Minimal surface; Pull back; Hodge *-operator; Winding number.

Riassunto. — *Un risultato di esistenza per sfere minimali in varietà con bordo.* Proviamo l'esistenza di una sfera minimale omotopicamente non banale in una 3-varietà con bordo, ottenuta da una 3-varietà compatta Riemanniana orientata con bordo, avente secondo gruppo di omotopia nullo, eliminando un aperto connesso.

Introduction

In this paper we study the existence of not homotopically trivial minimal spheres in 3-manifolds with boundary. In a celebrated paper Sacks-Uhlenbeck [4] have proved the existence of such minimal spheres in Riemannian manifolds without boundary, having nontrivial second homotopy group. In view of this result we consider a 3-manifold with boundary $\tilde{M}$ such that $\pi_2(\tilde{M})=0$, and, deleting an open connected subset of $\tilde{M}$, we construct another 3-manifold with boundary M, such that $\pi_2(M)\neq 0$. We prove the existence of a not homotopically trivial minimal sphere in M by following some ideas contained in [2], where the existence of a minimal sphere «enclosing an obstacle $\bar{\Omega}$ in $\boldsymbol{R}^3$» is proved. The main difference from [2] is that, since we are not working in a Euclidean space, we cannot use the Volume functional that in [2] gives the analytic expression of «enclosing $\bar{\Omega}$». We have therefore to use another functional we already introduced in [3], which provides an integral expression of the winding number in the setting of manifolds with boundary.

1. Minimal spheres in manifolds with boundary

Let $\tilde{M}$ be a compact Riemannian 3-manifold with smooth boundary $\partial\tilde{M}$, such that $\pi_2(\tilde{M})=0$. Let U be an open connected subset of $\mathrm{int}(\tilde{M})$ and let us consider $M:=\tilde{M}\setminus U$. We assume that there exists a lipschitzian retraction from a neighbourhood $\mathcal{O}$ of M in $\tilde{M}$ into M.

Given $p\in U=\tilde{M}\setminus M$, we will denote $X(M)=\{f\in H^1(S^2,M)\ s.t.\ W(f,p)\neq 0\}$ where $W(f,p)$ is defined as follows (see [3])

$$W(f,p):=\int_{S^2} f^*\varphi_p\,, \tag{1.1}$$

(*) Pervenuta all'Accademia il 9 ottobre 1989.

in which $*$ denotes the pull back of differential forms, $\varphi_p = * dG_p$, where $G_p = G(p,\cdot)$, G is the Green's function of the Laplacian in $\tilde{M}$, and $*$ denotes the Hodge $*$-operator. We call $W(f,p)$ the winding number of f around p. Let us recall here the main properties of the winding number functional. $W(f,p)$ is an integer number, independent of $p \in \tilde{M} \setminus M$, so that the definition of the class $X(M)$ does not depend on the choice of p. If f is a continuous map homotopic to a constant in $M \setminus \{p\}$, then $W(f,p) = 0$. $W(f,p)$ is continuous in f with respect to the strong topology of H^1 and satisfies the following inequality

$$|W(f,p)| \leqslant C \int_{S^2} |df|^2, \tag{1.2}$$

with $C = C(\tilde{M}, M)$ positive constant, independent of $f \in H^1(S^2, M)$. $X(M)$ is not closed under weak H^1-convergence, but $f^*\varphi_p$ presents a concentration phenomenon similar to that described by Brézis-Coron-Lieb [1] for the Volume functional. More precisely, if f_k, $f \in H^1(S^2, M)$ *s.t.* $f_k \rightharpoonup f$ in H^1, then, passing to a subsequence,

$$f_k^* \varphi_p \rightharpoonup f^* \varphi_p + \sum_{i=1}^{m} d_i \delta_{a_i}, \tag{1.3}$$

where $a_i \in S^2$, $d_i \in \mathbf{Z}$, δ_{a_i} denotes the Dirac measure concentrated in a_i and $\rightharpoonup$ denotes the weak convergence of measures.

THEOREM 1.1. *The infimum* $I = \inf_{f \in X(M)} \int_{S^2} |df|^2$ *is achieved.*

REMARK 1.2. From (1.2) and since for $f \in X(M)$ $W(f,p) \in \mathbf{Z} \setminus \{0\}$, I is positive.

If f_k is a minimizing sequence in $X(M)$, then there exists $f \in H^1(S^2, M)$ *s.t.*

$$f_k \rightharpoonup f \text{ in } H^1 \qquad f_k \to f \text{ a.e.} \tag{1.4}$$

From the weak lower semicontinuity of the L^2-norm it follows that

$$\int_{S^2} |df|^2 \leqslant I,$$

so that it is sufficient to prove that f belongs to the class $X(M)$. If in (1.3) $d_i = 0$ for each i, or if

$$\sum_{i=1}^{m} d_i = 0,$$

the thesis follows immediately using the test function 1. In order to treat the general case, let us see that if $f_k^* \varphi_p$ «concentrates» at some a_i, then f_k loses, at the limit, at least as much energy as I.

PROPOSITION 1.3. *Let* $f_k \in X(M)$ *verify* (1.4) *and let us suppose that* $d_i \neq 0$ *for some* i *in* (1.3). *Then, for* ρ *small enough,*

$$\liminf_k \int_{B_\rho(a_i)} |df_k|^2 \geqslant \int_{B_\rho(a_i)} |df|^2 + I, \tag{1.7}$$

where $B_\rho(a_i)$ *denotes the geodesic ball centered in* a_i *of radius* ρ.

PROOF. Fix $\rho > 0$ *s.t.* $a_j \notin B_{2\rho}(a_i)$ for $i \neq j$. Possibly passing to a subsequence, there exists

$$\lim_k \int_{B_\rho(a_i)} |df_k|^2.$$

For almost each $r<\rho$, there exists a subsequence (depending on r) $f_{k_j}=f_{k_j(r)}$ such that

$$\sup_j \int_{\partial B_r(a_i)} |df_{k_j(r)}|^2 < +\infty \tag{1.8}$$

and hence $f_{k_j(r)} \rightharpoonup f$ in $H^1(\partial B_r(a_i), M)$. Now, let $h_j^{(r)}$ be a minimizer of

$$m(S^2\setminus B_r(a_i), f_{k_j}) = \inf\left\{ \int_{S^2\setminus B_r(a_i)} |dv|^2 \,\middle|\, v\in H^1(S^2\setminus B_r(a_i), M),\ v-f_{k_j}|_{S^2\setminus B_r(a_i)} \in H_0^1 \right\}.$$

Possibly passing to a subsequence, we get, with a proof similar to that in [2], that

$$h_j^{(r)} \to h^{(r)} \qquad \text{in } H^1(S^2\setminus B_r(a_i), M), \tag{1.9}$$

where $h^{(r)}$ is a minimizer of

$$m(S^2\setminus B_r(a_i), f) = \inf\left\{ \int_{S^2\setminus B_r(a_i)} |dv|^2 \,\middle|\, v\in H^1(S^2\setminus B_r(a_i), M), v-f|_{S^2\setminus B_r(a_i)} \in H_0^1 \right\}.$$

It is easy to see that $m(S^2\setminus B_r(a_i), f) = m(B_r(a_i), f)$, so that

$$\int_{S^2\setminus B_r(a_i)} |dh^{(r)}|^2 \leqslant \int_{B_r(a_i)} |df|^2. \tag{1.10}$$

Let us consider

$$\tilde f_j^{(r)}(\sigma) = \begin{cases} f_{k_j(r)}(\sigma) & \text{if } \sigma\in B_r(a_i), \\ h_j^{(r)}(\sigma) & \text{if } \sigma\in S^2\setminus B_r(a_i), \end{cases} \tag{1.11}$$

$$|W(\tilde f_j^{(r)}, p)| = \left| \int_{B_r(a_i)} f_{k_j(r)} * \varphi_p + \int_{S^2\setminus B_r(a_i)} h_j^{(r)} * \varphi_p \right| \underset{j\to+\infty}{\longrightarrow} \left| \int_{B_r(a_i)} f^* \varphi_p + d_i + \int_{S^2\setminus B_r(a_i)} h^{(r)} * \varphi_p \right| \geqslant$$

$$\geqslant |d_i| - \left| \int_{B_r(a_i)} f^* \varphi_p \right| - C \int_{S^2\setminus B_r(a_i)} |dh^{(r)}|^2 \geqslant |d_i| - \left| \int_{B_r(a_i)} f^* \varphi_p \right| - C \int_{B_r(a_i)} |df|^2 > 0,$$

for r small enough, so that $W(\tilde f_j^{(r)}, p)\neq 0$ for r small enough and $j\geqslant j(r)$ big enough. Hence

$$I \leqslant \liminf_j \int_{S^2} |d\tilde f_j^{(r)}|^2 \leqslant \liminf_j \int_{B_r(a_i)} |df_{k_j(r)}|^2 + \int_{B_r(a_i)} |df|^2.$$

Then, for such r's we have

$$\lim_k \int_{B_\rho(a_i)} |df_k|^2 = \lim_j \left[\int_{B_r} |df_{k_j(r)}|^2 + \int_{B_\rho - B_r} |df_{k_j(r)}|^2 \right] \geqslant I + \int_{B_\rho} |df|^2 - 2\int_{B_r} |df|^2$$

and, passing to the limit for $r\to 0$, (1.7) follows. ■

From Proposition 1.3 and from the weak lower semicontinuity of the L^2-norm Corollary 1.4 immediately follows:

COROLLARY 1.4. *In the hypothesis of Proposition* 1.3, *we have that*

$$\liminf_k \int_{S^2} |df_k|^2 \geq I + \int_{S^2} |df|^2 .$$

LEMMA 1.5. *Let f_k satisfy the hypothesis of Proposition* 1.3 *and, in addition,*

$$\int_{S^2} |df_k|^2 \to I.$$

Then, with the previous notations, f is constant and $m=1$ in (1.3), *that is there exists a unique $a \in S^2$ s.t.*

$$f_k^* \varphi_p \rightharpoonup f^* \varphi_p + d\delta_a, \tag{1.12}$$

with $d \in \mathbf{Z}$ and

$$\int_{B_r(a)} |df_k|^2 \to I \qquad \forall r > 0. \tag{1.13}$$

PROOF. From Corollary 1.4 it follows that f is constant. Proposition 1.3 implies that there is exactly one $d_i \neq 0$ in (1.3), and hence (1.12) and (1.13) follow. ■

PROOF OF THEOREM 1.1. Let f_k be a minimizing sequence, so that we may assume that $f_k \rightharpoonup f$ in $H^1(S^2, M)$ for some $f \in H^1(S^2, M)$ and that (1.3) holds. As we have seen above, if $d_i = 0$ for each i in (1.3) the thesis follows immediately. Let us suppose that $d_i \neq 0$ for some i, and hence, by lemma 1.5, $f_k^* \varphi_p \rightharpoonup f^* \varphi_p + d\delta_a$, for some $a \in S^2$, $d \in \mathbf{Z}$. Let us consider the stereographic projection ψ from the north pole $N = -a$, denote $U_k = f_k \circ (\psi)^{-1}$, $U = f \circ (\psi)^{-1}$ and consider the concentration function

$$Q_k(t) = \sup_{z \in R^2} \int_{B_t(z)} |dU_k|^2 ,$$

which is continuous, nondecreasing and such that

$$Q_k(0) = 0, \qquad \lim_{t \to \infty} Q_k(t) = \int_{R^2} |dU_k|^2 = \int_{S^2} |df_k|^2 \geq I .$$

Hence, given $\delta \in]0, I[$, for each $k \in N$, there exists $t_k > 0$ such that $Q_k(t_k) = \delta$. Let $z_k \in R^2$ be such that

$$\int_{B_{t_k}(z_k)} |dU_k|^2 > \frac{\delta}{2}$$

and let us consider $\tilde{U}_k(z) = U_k(t_k z + z_k)$ and $\tilde{f}_k = \tilde{U}_k \circ \psi$. Notice that

$$\tilde{f}_k \in X(M), \qquad \int_{S^2} |d\tilde{f}_k|^2 = \int_{S^2} |df_k|^2 \to I.$$

Possibly passing to a subsequence, $\tilde{f}_k \rightharpoonup \tilde{f}$, $\tilde{f}_k \to \tilde{f}$ *a.e.*, $\tilde{f}_k^* \varphi_p \rightharpoonup \tilde{f}^* \varphi_p + \tilde{d}\delta_{\tilde{a}}$ for some $\tilde{f} \in H^1(S^2, M)$, $\tilde{a} \in S^2$, $\tilde{d} \in \mathbf{Z}$.

If $\tilde{d} \neq 0$ two cases occur: either $\tilde{a}$ is the north pole of ψ $(\tilde{a} = -a)$, or not.

In the second case, by Lemma 1.5, if we denote $\tilde{b} = \psi(\tilde{a})$, we have that

$\int_{B_1(\tilde{b})} |d\tilde{U}_k|^2 \to I$, but

$$\int_{B_1(\tilde{b})} |d\tilde{U}_k|^2 = \int_{B_{t_k}(t_k\tilde{b}+z_k)} |dU_k|^2 \leq Q_k(t_k) = \delta < I\,,$$

so that we get a contradiction.

In the first case, let ψ' denote the stereographic projection having $a = -\tilde{a}$ as north pole and let $\tilde{\tilde{U}}_k = \tilde{f}_k \circ (\psi')^{-1} = \tilde{U}_k \circ \psi \circ (\psi')^{-1}$. Then, by Lemma 1.5, $\int_{B_1(0)} |d\tilde{\tilde{U}}_k|^2 \to I$, but

$$\int_{B_1(0)} |d\tilde{\tilde{U}}_k|^2 = \int_{[\psi\circ(\psi')^{-1}]B_1(0)} |d\tilde{U}_k|^2 = \int_{R^2\setminus B_1(0)} |d\tilde{U}_k|^2 = \int_{R^2} |dU_k|^2 - \int_{B_{t_k}(z_k)} |dU_k|^2$$

and hence

$$\frac{\delta}{2} < \int_{B_{t_k}(z_k)} |dU_k|^2 = \int_{R^2} |dU_k|^2 - \int_{B_1(0)} |d\tilde{\tilde{U}}_k|^2 \to 0\,,$$

so that also the second case cannot occur and therefore $\tilde{d} = 0$, so that the weak limit $\tilde{f}$ of the new minimizing sequence $\tilde{f}_k$ belongs to $X(M)$. ■

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