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Logics that are both paraconsistent and paracomplete

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Logica matematica. — *Logics that are both paraconsistent and paracomplete.* Nota di NEWTON C. A. DA COSTA, presentata (*) dal Socio G. ZAPPA.

ABSTRACT. — The Author describes new systems of logic (called «nonalethic») which are both paraconsistent and paracomplete. These systems are connected with the logic of vagueness and with certain philosophical problems (e.g. with some aspects of Hegel's logic).

KEY WORDS: Logic; Paraconsistent; Paracomplete; Vagueness.

RIASSUNTO. — *Logiche paraconsistenti e paracomplete.* L'autore descrive nuovi sistemi di logica (detta «nonaletica») che sono al tempo stesso paraconsistenti e paracompleti. Questi sistemi sono connessi con la logica della vaghezza e con alcuni problemi filosofici (per esempio, con taluni aspetti della logica di Hegel).

1. INTRODUCTION

In a series of papers the author has presented a hierarchy of paraconsistent logics (see, for example, [1] and [3]). Loosely speaking, a paraconsistent logic is a logic in which a proposition and its negation can be both true. On the other hand, in [4] a hierarchy of paracomplete logics was introduced, that are «dual», in a precise sense, of some paraconsistent logics studied in [1] and [2]. A logic is called paracomplete if, according to it, a proposition and its negation can be both false. In this note the author describes a new hierarchy of logics which are simultaneously paraconsistent and paracomplete. The new logics to be introduced here, following a suggestion of F. Miro Quesada, we dub *nonalethic.*

Nonalethic logics are important in connection with vagueness, constructivity, and some philosophical issues (for instance, Hegel's logic seems to be nonalethic).

2. THE HIERARCHY N_i, $0 \leq i \leq \omega$

To begin, we describe a propositional logic, N_1. The primitive symbols of N_1 are the following: 1) Propositional letters (or propositional variables); 2) Connectives: $\supset$ (implication), & (conjunction), $\vee$ (disjunction), and $\neg$ (negation); the symbol $\sim$, for equivalence, is defined as usual; 3) Parentheses. The symbols and terminology are those of [5] with obvious adaptations and extensions.

We now formulate the postulates of N_1. The basic idea underlying this calculus is to construct a logic such that (1) when the principle of excluded middle holds, N_1 reduces to the calculus C_1 (see [1], [3]), and (2) when the law of contradiction is satisfied, we obtain P_1 (see [4]). Of course, when both principles are satisfied we obtain classical propositional logic.

(*) Nella seduta dell'11 marzo 1989.

DEFINITION 1. $A^0 \underset{\text{Def}}{=} \neg(A \& \neg A)$ and $A^* \underset{\text{Def}}{=} A \vee \neg A$.

Postulates of N_1:

$\supset_1$) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$

$\supset_2$) $A \supset (B \supset A)$

$\supset_3$) $\dfrac{A \quad A \supset B}{B}$

$\supset_4$) $((A \supset B) \supset A) \supset A$

$\&_1$) $A \& B \supset A$

$\&_2$) $A \& B \supset B$

$\&_3$) $A \supset (B \supset A \& B)$

$\vee_1$) $A \supset A \vee B$

$\vee_2$) $B \supset A \vee B$

$\vee_3$) $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$

I $A^* \& B^0 \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$

II $A^0 \& B^0 \supset (A \supset B)^0 \& (A \& B)^0 \& (A \vee B)^0 \& (\neg A)^0$

III $A^* \& B^* \supset (A \supset B)^* \& (A \& B)^* \& (A \vee B)^* \& (\neg A)^*$

IV $A^0 \supset (A \supset \neg\neg A) \& (A \supset (\neg A \supset B))$

V $A^* \supset (\neg\neg A \supset A)$

VI $A^0 \vee A^*$

The concepts of proof, deduction etc. are defined as in [5].

THEOREM 1. Adding the principle of contradiction, $\neg(A \& \neg A)$, to N_1 we obtain P_1.

THEOREM 2. Adding the principle of excluded middle, $A \vee \neg A$, to N_1 we get C_1.

THEOREM 3. Adjoining to N_1 the schemes $A \vee \neg A$ and $\neg(A \& \neg A)$ we obtain the classical propositional calculus.

THEOREM 4. If we set $-A \underset{\text{Def}}{=} \neg A \& A^0$, we have in N_1:

$$\vdash (A \supset B) \supset ((A \supset -B) \supset -A), \ \vdash A \supset (-A \supset B), \ \vdash A \& -A \supset B, \ \vdash A \supset --A.$$

THEOREM 5. The following schemes are not provable in N_1:

$$\neg(A \& \neg A), \quad A \vee \neg A, \quad (A \supset B) \supset ((A \supset \neg B) \supset \neg A), \quad A \supset (\neg A \supset B),$$

$$(A \sim \neg A) \supset B, \quad A \supset (\neg A \supset \neg B), \quad (A \sim \neg A) \supset \neg B, \quad \neg A \& (A \vee B) \supset B,$$

$$A \sim \neg\neg A, \quad (A \sim B) \supset (\neg A \sim \neg B), \quad (\neg A \sim \neg B) \supset (A \sim B).$$

THEOREM 6. (i) N_1 contains classical positive propositional logic; (ii) If the prime subformulas of $\Gamma \cup \{A\}$ are $A_1, A_2, \ldots, A_k$, then $\Gamma \vdash A$ in the classical proposisitional calculus iff Γ, A_1^0, A_1^*, A_2^0, $A_2^*, \ldots$, A_k^0, $A_k^* \vdash A$ in N_1.

THEOREM 7. N_1 is not decidable by finite logical matrices.

DEFINITION 2. Let e: $\mathcal{F} \to \{0,1\}$ be a function whose domain, $\mathcal{F}$, is the set of formulas of N_1. e is said to be an evaluation if we have:

i) If A is an axiom them $e(A) = 1$;
ii) If $e(A) = e(A \supset B) = 1$, then $e(B) = 1$;
iii) There exists a formula B such that $e(B) = 0$.

DEFINITION 3. Let Γ and $\Delta \neq 0$ be two sets of formulas. We say that Γ *excludes* Δ if $\Gamma \nvdash A$ for every A in Δ. Γ is said to be Δ*-saturated* if Γ excludes Δ, and for every B such that $\Gamma \nvdash B$, $\Gamma \cup \{B\}$ does not exclude Δ.

DEFINITION 4. Γ is *inconsistent* if $\Gamma \vdash A$ and $\Gamma \vdash \neg A$ for some formula A; otherwise Γ is *consistent*. Γ is said to be *trivial* if $\Gamma \vdash A$ for every formula A; otherwise, Γ is said to be *nontrivial.*

DEFINITION 5. A *valorization* is the characteristic function of a Δ-saturated set, for some set Δ. We define the notion of *valuation* as in [6].

THEOREM 8. If v is a valorization, then v is also an evaluation and a valuation.

THEOREM 9. Given any nontrivial set of formulas Γ, there exists a valorization v such that $v(A) = 1$ for every A in Γ.

As in [6], with the help of the concept of valorization we can define the notions of model of a set of formulas, of semantic consequence, and the symbol $\models$. We then have:

THEOREM 10. $\Gamma \vdash A$ iff $\Gamma \models A$ in N_1 (soundness and completeness theorem for N_1).

THEOREM 11. N_1 is decidable.

THEOREM 12. There are valorizations v such that $v(A) = v(\neg A) = 1$, for some formula A, and valorizations v' such that $v'(B) = v'(\neg B) = 0$, for some formula B.

Taking into account the calculi C_i, $0 \leqslant i \leqslant \omega$, of [1], and P_i, $0 \leqslant i \leqslant \omega$, of [4], we can easily construct a hierarchy N_i, $0 \leqslant i \leqslant \omega$, of nonalethic logics. This hierarchy has the expected properties, similar to those of C_i, $0 \leqslant i \leqslant \omega$, and of P_i, $0 \leqslant i \leqslant \omega$.

3. EXTENSIONS OF THE LOGICS N_i, $0 \leqslant i \leqslant \omega$.

Starting with the propositional calculi N_i, $0 \leqslant i \leqslant \omega$, it is not difficult to construct the corresponding first-order predicate calculi, with or without equality, and calculi of descriptions, and finally a family of corresponding set theories (as well as higher-order logics). In these *nonalethic* set theories, there exist sets that are «inconsistent», like Russell's set, and sets x which do not satisfy the formula $(\forall y)(y \in x \vee y \notin x)$ (see [2]).

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