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Minimal sets of generalized dynamical systems

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Matematica. — *Minimal sets of generalized dynamical systems* (*). Nota (**) di B. MESSANO e A. ZITAROSA, presentata dal socio E. DE GIORGI.

ABSTRACT. — We introduce generalized dynamical systems (including both dynamical systems and discrete dynamical systems) and give the notion of minimal set of a generalized dynamical system. Then we prove a generalization of the classical G. D. Birkhoff theorem about minimal sets of a dynamical system and some propositions about generalized discrete dynamical systems.

KEY WORDS: Generalized dynamical system; Birkhoff system; Minimal set.

RIASSUNTO. — *Insiemi minimali di sistemi dinamici generalizzati.* Introdotti i sistemi dinamici generalizzati (che comprendono tanto i sistemi dinamici quanto i sistemi dinamici discreti) e data la nozione di insieme minimale di un sistema dinamico generalizzato si provano, tra l'altro, un teorema che generalizza il classico teorema di G. D. Birkhoff relativo agli insiemi minimali di un sistema dinamico e alcune proposizioni concernenti i sistemi dinamici discreti generalizzati.

1. INTRODUCTION

Given a metric space D, it is said *dynamical system* (on D) a function f from $D \times R$ into D such that:

1) $f(x, 0) = x, \ \forall x \in D,$

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2) $f(f(x, t_1), t_2) = f(x, t_1 + t_2)$, $\forall x \in D$, $\forall t_1, t_2 \in R$,

3) *f is continuous with respect to both variables* (see [1,5,6]).

In theory of dynamical systems the notion of *minimal set* plays a very important role (see [1,5]).

In this respect it is fundamental the theorem, which we shall denote as «theorem of G. D. Birkhoff». Originally this theorem was proved by G. D. Birkhoff for the euclidean spaces (see [2]) and guarantees the existence of a minimal set contained in Y (see [1,5]), whatever the compact invariant set Y of a dynamical system be.

In this paper we call *generalized dynamical system* (*g.d.s.*) a pair $(S, (f_i)_{i\in I})$, where S is topological space and, for each $i \in I$, f_i is a function from S into itself.

A dynamical system f on a metric space D can be considered a generalized dynamical system going from the function f to the family of functions $(f_t)_{t\in R}$, where $f_t = f(\cdot, t)$, for each $t \in R$.

In section 2, after giving the notion of *positively invariant set, invariant set* and *minimal set* of a *g.d.s.* Σ, we prove (see (2.1)) that each closed compact positively invariant set Y of *g.d.s.* Σ contains a minimal element of the set of all closed positively invariant sets of Σ.

Moreover, after giving the definition of *Birkhoff system,* using theorem (2.1) we shall show (see (2.3)) that each closed compact positively invariant set of Birkhoff system Σ contains a minimal set of Σ.

Proposition (2.3) generalizes the theorem of G. D. Birkhoff.

We conclude section 2 giving two characterizations of the Birkhoff system $(S, (f_i)_{i\in I})$, when S is compact.

In section 3, we shall consider a topological space S satisfying the first axiom of countability, a continuous function f from S into itself and the generalized dynamical system $(S, (f^n)_{n\in N})$, where f^n is the nth iterate of f. Denoting the *g.d.s.* $(S, (f^n)_{n\in N})$ by (S, f) and, for each $x \in S$, the set of all cluster points of $(f^n(x))_{n\in N}$ by $\Omega(x)$, we shall prove (see (3.1)) some properties of $\Omega(x)$ and, moreover, we shall show (see (3.2)) that, if S is a Hausdorff compact space, the following statements are true:

1) *(S, f) is a Birkhoff system.*

2) *(S, f) is endowed with a minimal set.*

3) *A nonempy subset M of S is minimal set of (S, f) if and only if*

$$\Omega(x) = M \qquad \forall x \in M.$$

2. In what follows, S represents a topological space and $(f_i)_{i\in I}$ a family of functions from S into itself.

It is useful to denote the *g.d.s.* $(S, (f_i)_{i\in i})$ by Σ.

A nonempty subset X of S is said to be *positively invariant* (resp. *invariant*) *set* of Σ if $f_i(X) \subseteq X$ (resp. $f_i(X) = X$) for each $i \in I$.

Let us start by proving that:

(2.1) *For each closed compact positively invariant set Y of Σ, there exists a subset of Y which is a minimal element of the set of all closed positively invariant sets of Σ.*

Proof. Let us denote the set of all closed positively invariant sets of Σ by $\mathcal{P}$ and let us set:

$$\mathcal{P}_Y = \{X \in \mathcal{P} : X \subseteq Y\}.$$

We, immediately, observe that a subset of Y is a minimal element of $\mathcal{P}_Y$ if and only if it is a minimal element of $\mathcal{P}$.

Then, for proving the theorem it sufficies to demonstrate that $\mathcal{P}_Y$ is endowed with a minimal element.

Consequently, according to one property of compact spaces (see [4, C)]) it suffices to show that, for each nonempty subset $\mathcal{P}^*$ of $\mathcal{P}_Y$ such that:

$$\bigcap_{P \in \mathcal{P}^*} P \neq \emptyset, \tag{1}$$

it results that:

$$\bigcap_{P \in \mathcal{P}^*} P \in \mathcal{P}_Y. \tag{2}$$

Being:

$$f_i\left(\bigcap_{P \in \mathcal{P}^*} P\right) \subseteq f_i(P^*) \subseteq P^*, \qquad \forall P^* \in \mathcal{P}^*, \quad \forall i \in I,$$

it follows that:

$$f_i\left(\bigcap_{P \in \mathcal{P}^*} P\right) \subseteq \bigcap_{P \in \mathcal{P}^*} P, \qquad \forall i \in I;$$

the above inclusion and (1) imply (2).

The theorem is thus proved.

An invariant set of Σ is said *minimal set* of Σ if it is a minimal element of the set of all closed invariant sets of Σ.

We said that the generalized dynamical system Σ is a *Birkhoff system* if a subset M of S is a minimal set of Σ if and only if M is a minimal element of the set of all closed positively invariant sets of Σ.

Let us give a significative example of Birkhoff system.

Let G be an additive group and $(f_t)_{t \in G}$ be a family of functions from S into itself satisfying the following two conditions:

α) $f_0(x) = x$, $\forall x \in S$;

β) $f_{t_2}(f_{t_1}(x)) = f_{t_1+t_2}(x)$, $\forall x \in S$, $\forall t_1, t_2 \in G$.

Let us prove that:

γ) *A subset X of S is an invariant set of $(S, (f_t)_{t \in G})$ if and only if X is a positively invariant set of $(S, (f_t)_{t \in G})$.*

In fact, it suffices to observe that, if: $f_t(X) \subseteq X$, $\forall t \in G$, then: $X = f_0(X) = f_t(f_{-t}(X)) \subseteq f_t(X)$, $\forall t \in G$, whence: $X = f_t(X)$, $\forall t \in G$.

From γ) it follows that $(S, (f_t)_{t \in G})$ is Birkhoff system.

Consequently:

(2.2) *Each dynamical system is a Birkhoff system.*

From (2.1) the next proposition immediately follows which, according to (2.2), generalizes the theorem of G. D. Birkhoff:

(2.3) *If Σ is a Birkhoff system, each closed compact positively invariant set of Σ contains a minimal set of Σ.*

Let us now prove the following proposition:

(2.4) *If S is compact the following statements are equivalent:*

a) Each closed positively invariant set of Σ contains a closed invariant set of Σ.

b) Σ is a Birkhoff system.

c) Each closed positively invariant set of Σ contains a minimal set of Σ.

PROOF. *a)* $\Rightarrow$ *b)*. Let us set:

$$\mathcal{P} = \{P \subseteq S: P \text{ is a closed positively invariant set of } \Sigma\}.$$

If M is a minimal element of $\mathcal{P}$, then M is a minimal set of Σ. In fact, denoting a closed invariant set contained in M by Y, it obviously results that $Y = M$, and thus M is an invariant set of Σ.

Conversely, if M is a minimal set of Σ, then M is a minimal element of $\mathcal{P}$. In fact, if P is a subset of M that belongs to $\mathcal{P}$, denoting by Y a closed invariant set of Σ contained in P, it results that $Y = M$ from which $P = M$.

b) $\Rightarrow$ *c)*. The assertion trivially follows from (2.3).

c) $\Rightarrow$ *a)*. Trivial (1):

3. If the generalized dynamical system $(S, (f_i)_{i \in I})$ verifies one of the following hypothesis:

1) I is a singleton.

2) $I = N$ and, for each $i \in N$, f_i is the ith iterate f^i of f_1, then we say that $(S, (f_i)_{i \in I})$ is a *generalized discrete dynamical system* (*g.d.d.s.*); moreover, putting:

$$f = \begin{cases} \text{unique element of family } (f_i)_{i \in I} \text{ in case 1),} \\ f_1 \text{ in case 2),} \end{cases}$$

we shall denote $(S, (f_i)_{i \in I})$ by (S, f).

[If S is a metric space and f is continuous, the *g.d.d.s.* (S, f) is a discrete dynamical system on S (see [6, p. 223])].

Moreover, for each $x \in S$, set: $x_n = f^n(x)$ $\forall n \in N$, we denote the set of all cluster points of $(x_n)_{n \in N}$ by $\Omega(x)$.

Let us prove that:

(3.1) *If S satisfies the first axiom of countability and f is continuous, whatever $x \in S$ be, the following statements are true*:

μ) $\Omega(x)$ is a closed set; moreover, $\Omega(x)$ is a positively invariant set of (S, f) if it is nonempty.

(1) In proof of the implications *a)* $\Rightarrow$ *b)* and *c)* $\Rightarrow$ *a)* the hypothesis that S is compact is not necessary.

δ) $\Omega(x)$ is a compact invariant set of (S, f) if S is a Hausdorff space and there exists a compact positively invariant set Y of (S, f) such that $x \in Y$.

PROOF. Let us prove μ). Since $\Omega(x)$ is obviously closed, it suffices to show that, for each $y \in \Omega(x)$, $f(y) \in \Omega(x)$.

As S satisfies the first axiom of countability, there exists a subsequence $(x_{n_k})_{k \in N}$ of $(x_n)_{n \in N}$ such that: $\lim_k x_{n_k} = y$.

Then, being f continuous, we have: $\lim_k x_{n_k+1} = f(y)$, thus $f(y) \in \Omega(x)$.

μ) is thus proved. Then, to demonstrate δ) it suffices to show that: $\forall y \in \Omega(x)$, $\exists w \in \Omega(x): f(w) = y$.

According to the hypotheses on Y, there exists a strictly increasing sequence $(n_k)_{k \in N}$, of integers greater than 1, such that $(x_{n_k})_{k \in N}$ converges to y and, moreover, $(x_{n_k-1})_{k \in N}$ is convergent.

Let w be the limit of $(x_{n_k-1})_{k \in N}$. We obviously have that: $w \in \Omega(x)$, $f(w) = y$.

Hence the statement is proved [2].

In the end, by means of propositions (3.1) and (2.4) let us demonstrate the following proposition:

(3.2) *If S is a Hausdorff compact space satisfying the first axiom of countability and f is continuous, then the following statements are true*:

1) *(S, f) is a Birkhoff system.*

2) *(S, f) is endowed with a minimal set.*

3) *A nonempty subset M of S is a minimal set of (S, f) if and only if $\Omega(x) = M$ $\forall x \in M$.*

PROOF. According to δ) of (3.1) the statement *a*) of (2.4) is verified; consequently, from (2.4), 1) and 2) are satisfied.

Now, let us prove 3). If $\Omega(x) = M$ $\forall x \in M$, by means of (3.1) M is a closed invariant set of (S, f); moreover, if Y is a closed invariant set of (S, f) contained in M and $x \in Y$, then, obviously: $M = \Omega(x) \subseteq Y \subseteq M$, thus: $Y = M$.

Conversely, denoting a minimal set of (S, f) by M and an element of M by x, we have: $\Omega(x) = M$, for, according to (3.1), $\Omega(x)$ is a closed invariant set contained in M.

The theorem is thus proved.

[2] In the case in which S is an euclidean space (3.1) has been proved by J. P. La Salle (see [3, theorems 5.1 and 5.2]).

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