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Permutability of centre-by-finite groups

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Abstract. — Let G be a group and m be an integer greater than or equal to 2. G is said to be m-permutable if every product of m elements can be reordered at least in one way. We prove that, if G has a centre of finite index z, then G is $(1 + \lfloor z/2 \rfloor)$-permutable. More bounds are given on the least m such that G is m-permutable.

Key words: Centre-by-Finite Groups; Rewritable groups; Permutability.

Riassunto. — Sulla permutabilità dei gruppi centro-per-finito. Dato un gruppo G, si forniscono alcune limitazioni al minimo intero m tale che ogni prodotto di m elementi di G possa essere riordinato. In particolare, si prova che se il centro di G ha indice finito z allora $m \leq 1 + \lfloor z/2 \rfloor$.

Let G be a group. G will be said to have the property $P_m$, or to be m-permutable (for short $G \in P_m$) if every product of m elements in G can be rewritten (re-ordered), i.e. if for each m-tuple $(c_1, c_2, \ldots, c_m)$ of elements of G there exists a non trivial permutation $f$ on the set $\{1, 2, \ldots, m\}$, such that:

$$c_1 c_2 \ldots c_m = c_{f(1)} c_{f(2)} \ldots c_{f(m)}.$$

A group G will be said to be permutable if $G \in P_m$ for some m.

Such a property is a finiteness condition: a finitely generated torsion group (a.f.g. periodic semigroup, indeed, see [6]) is finite if and only if it is m-permutable for some m [2]. Curzio et al. characterized in [3] permutable groups: they are exactly the finite-by-Abelian-by-finite groups.

If G is a permutable group, a natural problem arises: which is the least integer m such that $G \in P_m$? Denote by $m(G)$ such an integer. In [5] it is shown that for finite groups,

$$\lim_{|G| \to \infty} m(G)/|G| = 0;$$

however the proof does not entail any efficient suggestion on $m(G)$ for relatively small groups, or for infinite permutable groups. On the contrary, this is given by the following result:

Proposition 1 [3]. If $|G'| = b$, then $G \in P_{b+1}$. If there exists an m-permutable subgroup N of G of index b, then $G \in P_{bm}$.

Prop. 1. Gives some immediate bounds for $m(G)$; e.g. the alternating group $A_5$ is immediately seen to be 20-permutable.

If the centre of G has a finite index, say $|G/Z(G)| = z$, (in the following z will always denote the index of $Z(G)$) then by Prop. 1. $G \in P_2$. But the existence of a non-trivial centre $Z(G)$ of G should naturally have a stronger effect on $m(G)$. In fact, we shall prove that $G \in P_{1 + \lfloor z/2 \rfloor}$ ($\lfloor b \rfloor$ will denote the greatest integer less than or equal to $b$). More generally, if $p$ is the least prime number dividing z, then $G \in P_{1 + \lfloor z/p \rfloor}$.

FINITE GROUPS.

If $G$ is a finite group of order $n$, then $G$ is trivially $n$-permutable. Longobardi and Maj (unpublished; reported in [1]) proved that $G$ has the property $P_{2(n+1)/3}$. In [4] Garzon and Zalcstein remarked that, if the finite group $G$ has a non-trivial centre of index $z$, then certainly $G \in P_{z+1}$.

Both results can be easily generalized, as follows: let $C_k = c_1 c_2 ... c_k$ be the product of a sequence of $k$ elements of $G$. Consider, for $i = 1, ..., [k/2]$, the cosets $c_{2i-1} Z(G)$, $c_{2i} Z(G)$, $c_{2i-2} c_{2i} 2(G)$. If the sequence $C_k$ cannot be rewritten, then all these cosets must be different: thus $k \leq 2z/3 + 1$. It immediately follows that $G$ belongs to $P_{[2z+1)/3]}$.

If $G$ is finite, however, another generalization can be easily obtained. Namely:

**Corollary 1.** Let $G$ be any finite group $G$ of order $n$ and let $p$ be the least prime integer dividing $n$; then $G$ belongs to $P_{(n/p)+1}$.

**Proof.** If the order of $G$ is divided by a prime $p$ (or $p^2$), then by Prop. 1, $G \in P_{n/p}$ (resp. $P_{n/p^2}$). Thus, if $G$ has an Abelian subgroup of order greater than or equal to 4, then $G \in P_{[n/2]}$. If this is not true, then $|G| \leq 6$: hence $G$ is Abelian or it is isomorphic to the symmetric group $S_3$, which belongs to $P_4$. Thus, generally $G \in P_{[n/2]+1}$.

Now, suppose that the least prime number dividing $n$ is $p > 2$. Then $|G'| \leq n/p$, and, again by Prop. 1, $G \in P_{n/p+1}$. #

Moreover, let us recall that any $p$-group $G$ of order $p^h$, has a normal Abelian subgroup of order $p^a$, with $a(a + 1) \geq 2b$ (see e.g. [7]). It follows immediately that the exponent $a$ must fulfill:

$$b \geq a \geq \left(-1 + \sqrt{1 + 8b}\right)/2.$$

Denote by $f(b)$, for each integer $b$, the least integer greater than or equal to

$$\left(-1 + \sqrt{1 + 8b}\right)/2.$$

Let now $n = p_1^{h_1} p_2^{h_2} ... p_j^{h_j}$ be the order of the finite group $G$. If $N$ is any $p_i$-Sylow subgroup of $G$ ($i = 1, ..., j$), then $N$ has an Abelian subgroup of order $p_i^{f(h_i)}$. Thus, by Prop. 1:

**Proposition 2.** Let $|G| = n = p_1^{h_1} p_2^{h_2} ... p_j^{h_j}$ and let $i$ be the index of the maximum power $p_i^{f(h_i)} (1 \leq i \leq j)$. Then $G$ has the permutation property $P_m$ with $m = 2n/p_i^{f(h_i)}$. #

One more bound can be obtained in general for the integer $m(G)$ if we look at the «permutable coverings» of $G$; namely:

**Proposition 3.** If there exists a finite family $\{H_i, 1 \leq i \leq b\}$ of groups such that $G \subseteq \bigcup_i H_i$, and $H_i \in P_m$, then $G \in P_m$, with $m = 1 + \sum_{i=1}^{b} (m_i - 1)$.

**Proof.** Consider a sequence $C_k$ of elements of $G$ which cannot be re-ordered: it cannot happen that $m_i$ different left factors of $C_k$ lie in the same subgroup $H_i$, for $1 \leq i \leq b$. But if $k \geq 1 + \sum_{i=1}^{b} (m_i - 1)$, this must happen for at least one $i$. Thus, we immediately get a re-ordering of the sequence. #
CENTRE-BY-FINITE GROUPS.

In the following, we want to give bounds for the integer $m(G)$, when $G$ is not necessarily finite, but the centre of $G$ has a finite index $z$.

Let us start with some general remarks on the sequences of elements of $G$. Let $C_k = c_1c_2\ldots c_k$ be the product of any sequence of elements: $k$ will be said the length of $C_k$; a subsequence $c_i\ldots c_j$ with $1 \leq i \leq j \leq k$ will be called a segment of $C_k$; a segment will be called a prefix if $i = 1$, a suffix if $j = k$.

**Lemma 1.** If one of the following holds, then the sequence $C_k$ of elements of $G$ can be re-ordered:

1) $c_i\ldots c_j \in (c_i\ldots c_k)Z(G)$, where $i<j$ or $b<g$;
2) $c_i\ldots c_j \in (c_i\ldots c_k)Z(G)$, where one segment is a proper prefix or suffix of the other;
3) $c_i\ldots c_j \in (c_i\ldots c_k)^{-1}Z(G)$, where $i<j$.

**Proof.** 1) Suppose $i<j$; we have:

$$c_1\ldots c_k = c_1\ldots c_i c_j\ldots c_k = c_1\ldots c_i c_j\ldots c_k x c_1\ldots c_i c_j\ldots c_k = (c_1\ldots c_i c_j\ldots c_k) = (c_1\ldots c_k c_j\ldots c_i)$$

for some $x \in Z(G)$.

2) trivially, after having cancelled the common elements, it remains a segment of the sequence $C_k$ which is central.

3) Since $c_i\ldots c_j = (c_i\ldots c_j)^{-1}x$, for some $x \in Z(G)$, then $(c_i\ldots c_j) = (c_i\ldots c_j)^{-2} x$ commutes with $(c_i\ldots c_j)$.

Let us now state two results on the sequences of a group $G$:

**Proposition 4.** Let $A$ be a (possibly trivial) subgroup of $G$ such that $A$ has a finite index $a$ in $G$. Every sequence $C_a$ contains a segment which is in $A$.

**Proof.** Consider all the prefixes of $C_a$:

$$c_1$$
$$c_1c_2$$
$$\ldots$$
$$c_1c_2\ldots c_a$$

They are a elements of $G$. If one of them is in $A$, then we have finished; otherwise, two elements are in the same (say: left) coset of $A$; that is there exist $y \in G, t, t' \in A$ such that:

$$c_1 \ldots c_i = yt$$
$$c_1 \ldots c_i \ldots c_j = y t'$$

Thus,

$$c_{i+1} \ldots c_j = t^{-1} t' \in A$$
**Proposition 5.** Let $A$ be a normal subgroup of $G$, of finite index $a$. If a sequence $C_{(a+1)/2}$ does not contain any segment lying in $A$, then it contains a segment which belongs to the coset $xA$, for every $x \in G \setminus A$ such that $x^2 \in A$.

**Proof.** Consider again the prefixes of $C_{(a+1)/2}$ and then multiply them by $x$, $x$ being any non-trivial element such that $x^2 \in A$, but $x \notin A$:

\[
\begin{array}{cccc}
  c_1 & c_1x \\
  c_1c_2 & c_1c_2x \\
  \cdots & \cdots \\
  \cdots & \cdots \\
  c_1c_2 \cdots c_{(a+1)/2} & c_1c_2 \cdots c_{(a+1)/2}x
\end{array}
\]

They are at least $a$ elements. Suppose that no segment of the sequence is in $A$.

If $c_1 \cdots c_i x \in A$, then we get

\[c_1 \cdots c_i x \in A \times ^{-1} = A x = x A.\]

If not, we must have one of the following cases:

i) $c_1 \cdots c_i$ and $c_1 \cdots c_i \cdots c_j$ belong to the same coset of $A$; this implies that a segment of the sequence is in $A$;

ii) $c_1 \cdots c_i x$ and $c_1 \cdots c_i \cdots c_j x$ belong to the same coset of $A$; this implies again that:

\[c_{i+j} \cdots c_{j} x \in A \times ^{-1} = A ;\]

iii) $c_1 \cdots c_i$ and $c_1 \cdots c_i \cdots c_j x$ belong to the same coset of $A$ (we may suppose $j > i$ without loss of generality); then:

\[c_{i+j} \cdots c_{j} x \in A \times ^{-1} = A x = x A . \]

If $G$ is finite, then we can choose $A = \{1\}$.

**Corollary 2.** Let $G$ be a finite group.

Every sequence $C_n$ of elements of $G$ contains a segment which is equal to 1.

If a sequence $C_{(a+1)/2}$ does not contain any segment equal to 1, then it contains a segment equal to $x$, for every $x \in G$ such that $x^2 = 1, x \neq 1$.

Let now $W = \{xZ(G) \in G/Z(G) : x \notin Z(G) \text{ and } x^2 \in Z(G)\}$; say $|W| = w$.

**Proposition 6.** In a sequence which cannot be re-ordered, there cannot be more than $[(1 + w)/2]$ prefixes (suffixes) belonging to different cosets in $W$.

**Proof.** Let $a$, $ab$ be two different prefixes in a sequence $C_k$, which cannot be re-ordered. Suppose that $aZ(G)$ and $abZ(G)$ belong to $W$. One can easily see that $bab \in aZ(G)$ and $baZ(G) \in W$. Suppose that $ba$ belongs to the same coset with a prefix of $C_k$ and say:

\[a = c_1 \cdots c_i, \quad b = c_{i+1} \cdots c_j, \quad ba \in c_1 \cdots c_j Z(G).\]

If $b = i$, then $b \in Z(G)$. 

\[^\#\]
If $b > j$, then say $c_{j+1} \ldots c_b = c$; we get:

$$ba \in abcZ(G), \quad bab \in abcbZ(G), \quad bcb \in Z(G)$$

and $b$ commutes with $c$.

If $b < i$, then say $c_{i+1} \ldots c_i = c$; we get:

$$bac \in aZ(G) = babZ(G), \quad c \in bZ(G),$$

and again $b$ commutes with $c$.

If $i < b < j$, then say $c_{i+1} \ldots c_b = c$ and $c_{b+1} \ldots c_j = d$; we get:

$$ba \in acZ(G) \quad \text{and} \quad bad \in abZ(G);$$

hence

$$c \in b^{-1}Z(G) \quad \text{and} \quad d \in (ba)(ab)Z(G) = ba^2bZ(G) = b^2Z(G);$$

thus $c$ and $d$ commute.

All these lead to a contradiction. Thus, it must be $h = j$, that is $ba \in abZ(G)$, whence $b \in abaZ(G) = b^{-1}Z(G)$ and $bZ(G) \in W$. By similar arguments, it can be seen that $b$ cannot be in the same coset with any prefix of the sequence: in fact, if we suppose that $b \in c_1 \ldots c_kZ(G)$, then trivially $g \neq i, j$; also, if $g < i$ or $i < g < j$, then we can commute $a$ and $b$; at last, if $g > j$, then we get $b \in abxZ(G) = baxZ(G)$, where $x$ is the segment $c_{j+1} \ldots c_j$, which is immediately seen to belong to $a^{-1}Z(G) = aZ(G)$, thus allowing a re-ordering of the sequence.

Now, let $a_1, a_1a_2, \ldots, a_1a_2\ldots a_b$ be the prefixes of $C_k$, which belong to $W$. We have shown that, for $f = 1, \ldots, b$, the cosets of $a_2 \ldots a_1a_2 \ldots a_b$ (or correspondingly of $a_2 \ldots a_1$) are elements of $W$, but do not correspond to any prefix of $C_k$. All of them must be trivially different from each other, or the sequence could be re-ordered; thus they are exactly $h - 1$.

Hence, it must be $b + h - 1 \leq w$ and $b \leq (1 + w)/2$. #

**Theorem 1.** Let $G$ be a group. If the centre $Z(G)$ has a finite index $w$, then $G \in P_{[2w] + 1}$.

**Proof.** Let $C_k$ be a sequence which cannot be re-ordered; consider all the cosets of $Z(G)$ containing the prefixes or their inverses:

<table>
<thead>
<tr>
<th>$c_1$</th>
<th>$c_i^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1 c_2$</td>
<td>$(c_1 c_2)^{-1}$</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>$c_1 \ldots c_k$</td>
<td>$(c_1 \ldots c_k)^{-1}$</td>
</tr>
</tbody>
</table>

An immediate consequence of Lemma 1. is that all cosets of the elements in the left (or in the right) column are different from each other and that an element in the left column is in the same coset with one in the right column if and only if that coset belongs to $W$.

If these are exactly $h$, then we get $2k - b$ different cosets.
Also, none of the present cosets can be $Z(G)$ itself. Hence $k$ must fulfil the condition:

$$2k - b \leq n/z - 1 - (w - b),$$

which by Prop. 6, implies:

$$2k \leq n/z - (w - 2b + 1) \leq n/z.$$

Thus, if $m \geq 1 + [n/2z]$, then $G$ must be $m$-permutable. 

**Theorem 2.** Let $G$ be a group; suppose that its centre has a finite index $z$ and let $p$ be the least prime number dividing $z$. Then $G \in P_{[z/(p-1)]+1}$.

**Proof.** Consider the following cosets of $Z(G)$, for a product $C_k$ which cannot be re-ordered:

- $c_1Z(G)$
- $c_1c_2Z(G)$
- $\ldots$
- $c_1\ldots c_kZ(G)$

All of them are trivially different from $Z(G)$, since every exponent is prime to the order of the corresponding prefix, and no prefix must be central. A similar argument shows that it cannot be:

$$(c_1 \ldots c_i)^{q}Z(G) = (c_1 \ldots c_i)^{q+1}Z(G).$$

In fact, if it were true, suppose $j > i$, we would have that, for some integer $q$,

$$c_1 \ldots c_j \in (c_1 \ldots c_i)^{q}Z(G).$$

This implies that the sequence could be re-ordered, as a segment is in the same coset of $Z(G)$ with an adjacent one.

Thus, it must be:

$$k(p - 1) \leq n + z - 1,$$

and this proves the result. 

**References**