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Sharp regularity theory for second order hyperbolic equations of Neumann type


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Equazioni a derivate parziali. — Sharp regularity theory for second order hyperbolic
equations of Neumann type(*). Nota(**) di IRENA LASIECKA e ROBERTO TRIGGIANI,
presentata dal Corrisp. R. CONTI.

ABSTRACT. — This note provides sharp regularity results for general, time-independent, second order
hyperbolic equations with non-homogeneous data of Neumann type.

KEY WORDS: Hyperbolic partial differential equations.

RIASSUNTO. — Regolarità delle soluzioni di equazioni differenziali iperboliche del secondo ordine con dati
al contorno di tipo Neumann. Si danno risultati di regolarità delle soluzioni del problema misto per equazioni
da derivate parziali del secondo ordine di tipo iperbolico, con dato non omogeneo sulla frontiera di tipo
Neumann.

1. Regularity problem, preliminaries, and statement of main results

Let \( x > 0 \) be a scalar positive variable, \( t \) be a real variable, and \( y = [y_1, ..., y_{n-1}] \) be an \((n-1)\)-dimensional vector with real components. In symbols: \( x \in \mathbb{R}_+^1; \ t \in \mathbb{R}_+^1; \ y \in \mathbb{R}_{n-1}^1 \). Let

\[
\begin{align*}
\Omega &\equiv \mathbb{R}_+^1 \times \mathbb{R}_{n-1}^1, \\
\Gamma &\equiv \mathbb{R}_{n-1}^1 = \Omega|_{x=0}
\end{align*}
\]

be, respectively, an \( n \)-dimensional half-space \( \Omega \) with boundary \( \Gamma \). On \( \Omega \) we consider the
second order differential operator

\[
P(x,y; D_t, D_x, D_y) = -aB^2 + \sum_{i,j=1}^{n-1} a_{ij} D_y D_x + D_x^2
\]

with space-dependent, but time-independent coefficients

\[
a = a(x,y), \quad a_{ij} = a_{ji}(x,y), \quad i = 1, ..., n; \ [x,y] \in \Omega, \ j = 1, ..., n - 1
\]
satisfying the symmetricity conditions \( a_{ij} = a_{ji}, i, j = 1, ..., n - 1 \). Here and throughout
we use the notation

\[
D_t = \frac{1}{\sqrt{1-1}} \frac{\partial}{\partial t}; \quad D_x = \frac{1}{\sqrt{1-1}} \frac{\partial}{\partial x}; \quad D_y = \frac{1}{\sqrt{1-1}} \frac{\partial}{\partial y}; \quad \text{etc.}
\]

On \( \Gamma \), the boundary of the half-space \( \Omega \), we consider the first order operator

\[
B(y; D_x, D_y) = D_x + \sum_{j=1}^{n-1} b_j D_y
\]

with space-dependent, but time-independent coefficients

\[
b_j = b_j(y), \quad y \in \Gamma.
\]

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Superiore, Pisa, nel luglio 1985; e, inoltre, all'IFIP Conference, Santiago di Campostela, Spagna, nel luglio
1987.
The present paper investigates regularity properties of the solution \( u(t,x,y) \) of the following second order hyperbolic mixed problem with Neumann boundary conditions

\[
\begin{aligned}
(a) & \quad P(x,y; D_x, D_y) u = f(t,x,y) \quad \text{on } \Omega, \quad t > 0, \\
(b) & \quad B(y; D_x, D_y) u = g(t,y) \quad \text{on } \Gamma, \quad t > 0, \\
(c) & \quad u|_{t=0} = u_0; \quad D_t u|_{t=0} = u_1 \quad \text{on } \Omega, \quad t = 0,
\end{aligned}
\]  

(1.6)

at least for a few specific fundamental function spaces for \( f \) and \( g \). Other classes of function spaces will be examined in a subsequent paper [13]. Generally, we are interested in the continuity of the map from the data \((u_0, u_1, f, g)\) in preassigned function spaces (possibly, subject to compatibility conditions) into the solution \( u, u_t, ... \) and possibly its trace \( u|_{\Gamma}, ... \) in suitable (optimal) function spaces. Throughout the paper, problem (1.6) will be subject to the following assumptions:

(i) the coefficients \( a, a_{ij}, a_{nj} \) of \( P \) and \( b_j \) of \( B \) are assumed real, time independent, sufficiently smooth in the space variables, and constant outside a compact set \( \mathscr{K}_0 \) of \( R_1^+ \times R_1^- = \Omega; \)

(ii) the boundary \( \Gamma (x = 0) \) is non-characteristic for \( P \) and \( P \) is «regularly hyperbolic with respect to \( t \); i.e. the characteristic polynomial of \( P \),

\[
(1.7a) \quad p(x,y; \xi, \eta) = -a \xi^2 + \sum_{i,j=1}^{n-1} a_{ij} \xi_i \eta_j + 2 \xi \sum_{j=1}^{n-1} a_{nj} \eta_j + \xi^2
\]

(1.7b) \quad \quad = -a \xi^2 + \left( \xi + \sum_{j=1}^{n-1} a_{nj} \eta_j \right)^2 + \sum_{i,j=1}^{n-1} a_{ij} \xi_i \eta_j - \left( \sum_{j=1}^{n-1} a_{nj} \eta_j \right)^2

has two real and distinct roots in \( \tau \), for \( (x,y) \in \Omega \) and \( (\xi, \eta) \) on the unit sphere \( \xi^2 + |\eta|^2 = 1 \), where \( |\eta|^2 + \sum_{j=1}^{n-1} \eta_j^2 \). If we consider \( \eta = 0 \) and \( \xi = 1 \), this requirement yields the condition

\[
(1.8) \quad \min a(x,y) > 0 \quad \text{in } \Omega;
\]

moreover, if we consider the points of the unit sphere in \((\xi, \eta)\) which lie also on the hyperplane \( \xi + \sum_{j=1}^{n-1} a_{nj} \eta_j = 0 \), this requirement yields the necessary condition, which is plainly also sufficient, that the quadratic form in \( \eta \)

\[
(1.9) \quad d(x,y; \eta) = a^2(x,y) \left\{ \sum_{i,j=1}^{n-1} a_{ij} (x,y) \eta_i \eta_j - \left( \sum_{j=1}^{n-1} a_{nj} (x,y) \eta_j \right)^2 \right\}
\]

(independent of \( \xi \)) be positive definite

\[
(1.10) \quad d(x,y; \eta) > 0 \quad \text{in } \Omega, \quad |\eta|^2 \neq 0;
\]

(iii) the first order operator \( \widetilde{D}_x \) defined by

\[
(1.11) \quad \widetilde{D}_x = D_x + \sum_{j=1}^{n-1} a_{nj} (x,y) D_{\eta_j}
\]
restricted on the boundary $I'$, coincides with $B$; i.e.
\begin{equation}
B = \overline{D}_{|x=0} \quad \text{i.e.} \quad b_j(y) \equiv a_{w}(0, y), \quad j = 1, \ldots, n - 1.
\end{equation}

The following results are known and provide the a-priori regularity needed in the subsequent development.

**Lemma 1.1.** Let $u_0 = u_1 = 0$ in (1.6c) and let $0 < T < \infty$.

\( a \) Let $g \equiv 0$ and $f \in L_1(0, T; L^2(\Omega))$ in (1.6). Then
\begin{align*}
\begin{aligned}
&u \in C([0, T]; H^1(\Omega)) \\
u_t \in C([0, T]; L^2(\Omega))
\end{aligned}
\end{align*}
(a fortiori $u \in H^1([0, T] \times \Omega)$) continuously.

\( b \) Let $f \equiv 0$ and $g \in L_2(0, T; L^2(\Omega))$ in (1.6). Then (i)
\begin{align*}
\begin{aligned}
&u \in C([0, T]; H^{1/2}(\Omega)) \\
u_t \in C([0, T]; H^{-1/2}(\Omega))
\end{aligned}
\end{align*}
(a fortiori $u \in H^{1/2}([0, T] \times \Omega)$).

Trace theory applied to Lemma 1.1a) then gives
\begin{equation}
\begin{aligned}
f \in L_1(0, T; L^2(\Omega)) \\
g = 0 \\
u_0 = u_1 = 0
\end{aligned}
\end{equation}
continuously.

**Main Theorem 1.2.** Let $g = 0$ and $u_0 = u_1 = 0$ and let $f \in L_2(Q^+)$, $Q^+ = R^i \times \Omega$.

Then,
\( a \) If $\Sigma^+ = R^i \times I^+$, the trace $u|_{\Sigma}$ of the solution to (1.6) satisfies $u|_{\Sigma} \in H^{3/5}(\Sigma^+)$ continuously: there is a constant $C > 0$ independent of $f$ such that
\begin{equation}
\|u|_{\Sigma}\|_{H^{3/5}(\Sigma^+)} \leq C \|f\|_{L_2(Q^+)}.
\end{equation}

\( b \) In the special cases where the coefficients $a_{ij}$, $i, j = 1, \ldots, n - 1$; $a_{ij}$, $j = 1, \ldots, n-1$ either do not depend on $x$, or else do not depend on $y$, then $u|_{\Sigma} \in H^{2/3}(\Sigma^+)$ continuously: there is a constant $C > 0$ independent of $f$ such that
\begin{equation}
\|u|_{\Sigma}\|_{H^{2/3}(\Sigma^+)} \leq C \|f\|_{L_2(Q^+)}.
\end{equation}

**Remarks 1.1.** (i) The general case (1.14) represents an improvement by $\leq 1/10$ ($1/2 + 1/10 = 3/5$) in the space regularity of the trace over (1.13).

(ii) Addition of a first order differential operator to $P$ does not affect the results. \( \Box \)

A second main result of this paper is the following

\( ^{(\text{I})} \) Lions-Magenes, vol. II, p. 120 provide only $L_2(0, T; \cdot)$; but this can be improved to $C([0, T]; \cdot)$ with the same space regularity, as e.g. in [3], [11].
Main Theorem 1.3. Let \( f = 0, \ u_0 = u_1 = 0, \) and \( g \in L_2(\Sigma_+)\). Then, continuously, for any \( \varepsilon > 0 \):

\[
\begin{align*}
(1.16) & \quad \begin{cases} 
  \nu \in H^{3/5-\varepsilon}(Q_+)(\text{improvement by almost } \xi^{1/10} \text{ over Lemma 1.1b}) \\
  \text{AND} \\
  u|_\Sigma \in H^{1/3-\varepsilon}(\Sigma_+) 
\end{cases} \\
(1.17) & \quad \begin{cases} 
  \nu \in H^{3/5-\varepsilon}(Q_+) \\
  \text{AND} \\
  u|_\Sigma \in H^{1/3}(\Sigma_+) 
\end{cases}
\end{align*}
\]

b) In the special case where the coefficients \( a_{ij}, a_{ii}, i,j = 1,\ldots,n-1 \) either do not depend on \( x \), or else do not depend on \( y \), then

\[
\begin{align*}
(1.18) & \quad \begin{cases} 
  u \in H^{3/2}(Q_+) \\
  \text{AND} \\
  u|_\Sigma \in H^{1/3}(\Sigma_+) 
\end{cases} \\
(1.19) & \quad \begin{cases} 
  u \in H^{3/4}(Q_+) \\
  \text{AND} \\
  u|_\Sigma \in H^{1/3}(\Sigma_+) 
\end{cases}
\end{align*}
\]

Remarks 1.2. (i) For \( \dim Q \geq 2 \) and the Laplacian case, one can show that \( u \notin H^{3/4+\varepsilon}(Q), \ \forall \varepsilon > 0 \) [5], [12].

(ii) Result (1.17) is a regularity result. Trace theory applied to interior regularity (1.16) gives only \( H^{3/5-1/2-1/10}(\Sigma) \), a result worse than (1.17) by \( \xi^{1/10} \). Similarly, trace theory applied to (1.18) gives \( H^{3/5-1/2-1/6}(\Sigma) \), a result worse than (1.19) by \( \xi^{1/6} \).

(iii) The regularity in (1.18)-(1.19) coincides with that proved directly, by eigenfunction expansions, for the Laplacian \( \Delta \) on a sphere = \( \Omega \) [3].

(iv) Direct computations, by eigenfunction expansions, with the Laplacian on a parallelepiped \( \Omega \) produce \( u \in H^{3/4+\varepsilon}(Q), \varepsilon > 0 \) and \( u|_\Sigma \in H^{1/3}(\Sigma_+) \) [3].

The proofs of Theorems 1.2 and 1.3 are very lengthy and technical and are to be found in [12].

References


