Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

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## On a class of variational integrals over BV varieties

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **83** (1989), n.1, p. 101–108.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1989_8_83_1_101_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1989.

*Atti Acc. Lincei Rend. fis.*
*(8), LXXXIII (1989), pp. 101-108*

**Analisi funzionale.** — *On a class of variational integrals over BV varieties* (*). Nota di PRIMO BRANDI (**) e ANNA SALVADORI (***), presentata (****) dal Socio Straniero L. CESARI.

ABSTRACT. — We present here our most recent results ([1*def*]) about the definition of non-linear Weiertrass-type integrals over BV varieties, possibly discontinuous and not necessarily Sobolev's.

KEY WORDS. — Calculus of variations on BV varieties; Weierstrass integrals; Burkill-Cesari integrals.

RIASSUNTO. — *Su una classe di integrali variazionali per varietà BV.* In questa nota presentiamo brevemente alcuni nostri recenti risultati ([1*def*]) relativi alla definizione di integrale non-lineare alla Weierstrass su varietà *BV*, possibilmente discontinue e non di Sobolev.

## 1. INTRODUCTION

In [1*ab*] Cesari established a very general axiomatization concerning extensions of Burkill's integral on set functions. Namely, he introduced a concept of quasi-additivity for set functions garanteeing the existence of a limit, now called the Burkill-Cesari integral. About the non-linear integral $I = \int_T F(p,q)$ over a variety $T$, Cesari considered the set function $\Phi(I) = F(T(\omega(I)), \varphi(I))$, where $\omega(I)$ is a choice function, i.e. $\omega(I) \in I$, and $\varphi$ is a set function. He proved that if $T$ is any continuous parametric mapping and $\varphi$ is quasi-additive and $BV$, then also $\Phi$ is quasi-additive and $BV$. In other words, the non-linear transformation $F$ preserves quasi-additivity and bounded variation. Then the integral $I$ is defined by the Burkill-Cesari process on the function $\Phi$, and $I$ is thus defined as a Weierstrass-type integral.

Later, many authors studied this integral, both in the parametric and in the non-parametric case, for continuous varieties, and framed in this theory many of his properties (see [6] for a survey). Only in the case that $F$ does not depend on the variety, i.e. it is of the type $F(q)$, then the sole concept of quasi-additivity permits the extension of $I$ over $BV$ curves and surfaces, not necessarily continuous, not Sobolev's.

In the last years, in force of a new condition of quasi-additivity type, we have extended the definition of $I$ over $BV$ curves or varieties, not necessarily continuous and not Sobolev's, for complete integrands $F(p, q)$ (see [1*bcdef*]). Here we present our most recent results on this subject which will appear in [1*def*] with details and proofs.

First we replaced the term $T(\omega(I))$, in the definition of $\Phi(I)$, with a set function $P(I)$ whose values are in a metric space $K$, while we take for $\varphi(I)$ a set function whose values are in a uniformly convex Banach space $X$ and $F: K \times X \to E$, with $E$ real Banach space. In order to garantee the existence if the integral $I$ for $BV$ transformations $T$, we proposed in [1*d*] a condition on the pair of set function $(P,\varphi)$, which is of quasi-

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additivity-type, that we called $\Gamma$-quasi-additivity. This condition reduces to the quasi-additivity on $\varphi$ when $P$ is the usual set function $T(\omega(I))$ and $T$ is continuous. In this new situation, we have been able to prove that, if $(P, \varphi)$ is $\Gamma$-quasi-additive and $\varphi$ is $BV$, then still $\Phi(I) = F(P(I), \varphi(I))$ is quasi additive and $BV$. Thus the integral $I$ is still defined by the Burkill-Cesari process on the set function $\Phi$, and $I$ is still a Weierstrass-type integral even for $T$ only $BV$, possibly discontinuous.

Note that the new condition on $(P, \varphi)$ is weaker than the couple of assumptions: continuity on $T$ and quasi-additivity on $\varphi$. Moreover, it takes advantage of the power of the quasi-additivity-type properties to extend $I$ over $BV$ curves and varieties, for integrands of the type $F(p, q)$, both in the parametric and in the non-parametric case (see many applications in [1 *def*]).

We wish also to mention that even in this more general setting, we prove that the integral $I$ admits a Lebesgue-Stieltjes integral representation ([1*d*])

$$I = \int_A F(\pi(a), (d\mu/d\|\mu\|)(a))\, d\|\mu\|$$

in terms of the vectorial measure $\mu$ related to $\varphi$, its total variation $\|\mu\|$, and Radon-Nikodym derivative $d\mu/d\|\mu\|$, as in the previous work of Cesari [3*b*] in Euclidean spaces and in our successive extension to abstract spaces, always for continuous varieties $T$ (see [6] for a survey).

In the non-parametric case (see [1*e*]) the integral $I = \int_T f(t, p, q)$ is transformed into a suitable parametric integral in the manner of McShane, with the integrand $F(t, p; l, q)$ defined by $F(t, p; l, q) = l f(t, p, q/l)$ for $l > 0$ and $F(t, p; 0, q) = \lim_{l \to 0^+} F(t, p; l, q)$. Then the set function $\Phi$ becomes $\Phi(I) = \lambda(I) f(p(I), \varphi(I)/\lambda(I)) = F(p(I); \lambda(I), \varphi(I))$. Thus the existence result is still given in terms of $\Gamma$-quasi-additivity. Now the representation of $I$ in terms of Lebesgue-Stieltjes integral becomes

$$I = \int_A f(\pi(a), (d(\nu, \mu)/d\|(\nu, \mu)\|)(a))\, d\|(\nu, \mu)\|$$

where $\mu$ is the vectorial measure related to $\varphi$, $\nu$ is the real measure related to $\lambda$ and $\|(\nu, \mu)\|$ is the total variation of the measure $(\nu, \mu)$.

Furthemore, in this non-parametric situation, we proved a Tonelli-type inequality comparing $I$ with a corresponding Lebesgue-Stieltjes integral, namely,

$$I \geqslant \int_A f(\pi(a), (\partial\mu/\partial\nu)(a))\, d\nu\,,$$

where $\partial\mu/\partial\nu$ is a derivative of the Radon-Nikodym type, and the equality sign holds if and only if the set function $\varphi$ is absolutely continuous with respect to the set function $\lambda$. Note that, if $\varphi$ is absolutely continuous with respect to $\lambda$, then $\partial\mu/\partial\nu$ reduces to the usual Radon-Nikodym derivative $d\mu/d\nu$.

We wish to mention that in proving this last result, as in the proof of the representation theorem, we make use of a connection between the Burkill-Cesari process and the convergence of martingales, a connection which we already pointed out in previous papers (see [6] and the quoted papers [1*def*]).

Finally in [1*f*] we dealt with the problem of the lower semicontinuity for the integral $I$, both in the parametric and in the non-parametric case. A first result on this subject has been given by Warner ([7*a*]) who proposed a lower semicontinuity theorem which contains, as applications, the classical theorems by Tonelli and Turner. Successively, in [1*b*], we have presented a modified version of such a result in order to widen the field of applications. Again with the same spirit, in [1*f*] we present first an abstract lower semicontinuity theorem, in terms of a suitable global convergence on the sequence $((p_n, \varphi_n))_n$, defined in the same spirit of the $\Gamma$-quasi-additivity and therefore inspired to Cesari's concept of quasi additivity. Then we show that in a number of applications this convergence is implied by the $L_1$-convergence of equi $BV$ varieties.

We further note here that independent work on the calculus of variations for $BV$ varieties, possibly discontinuous, possibly not Sobolev, has been done by Cesari, Brandi, and Salvadori [4*abc*], in connection with the Serrin integral [5] associated to the usual Lebesgue integral, and in view of many different applications.

## 2. The Weierstrass-type integral.

Let $(A, \mathcal{G})$ be a topological space, we denote by $\{I\}$ a family of subsets of $A$ that we call intervals. A finite system $D = [I_1, \ldots, I_N]$ is a finite collection of non-overlapping intervals, i.e. $I_i^0 \neq \emptyset$ and $I_i^0 \cap \bar{I}_j = \emptyset$, $i \neq j$, $i, j = 1, \ldots, N$ (where $I^0$ and $\bar{I}$ denote the $\mathcal{G}$-interior and the $\mathcal{G}$-closure of $I$ respectively). Let $(T, \gg)$ be a directed set and let $(D_t)_{t \in T}$ be a given net of finite systems.

Let $(K, d)$ be a metric space, $X$ be a uniformly convex Banach space and $E$ be a real Banach space. We consider functions: $F: K \times X \to E$, $p: \{I\} \to K$, $\varphi: \{I\} \to X$ and denote by $\Phi: \{I\} \to E$ the set function defined by

$$\Phi(I) = F(p(I), \varphi(I)) .$$

The function $\Phi$ is said to be *Burkill-Cesari integrable* ($BC$-integrable) ([3*a*]) if the limit $\lim_T \sum_{I \in D_t} \Phi(I)$ exists. Following Cesari ([3*a*]), the $BC$-integral of the function $\Phi$, when it exists, will be called the *parametric Weierstrass-integral of the Calculus of Variations* ($W$-integral) and denoted by $BC \int_A F(p, \varphi)$.

A function $\varphi$ is said to be *quasi-additive* (q.a.) (cfr. Cesari [3*a*]) over $M \subset A$, if (q.a.) *given* $\varepsilon > 0$ *there exists* $t_1 = t_1(M, \varepsilon)$ *such that, for every* $t_0 \gg t_1$ *there exists* $t_2 = t_2(M, \varepsilon, t_0)$ *such that, if* $t \gg t_2$ *then*

$$\text{i)}\ \sum_I s(I, M) \left\| \sum_J s(J, I)\, \varphi(J) - \varphi(I) \right\| < \varepsilon ,$$

$$\text{ii)}\ \sum_J s(J, M) \left[ 1 - \sum_I s(J, I)\, s(I, M) \right] \| \varphi(J) \| < \varepsilon ,$$

*where* $D_{t_0} = [I]$, $D_t = [J]$ *and* $s(H, L) = 1$ *when* $H \subset L$, $s(H, L) = 0$ *otherwise.*

The function $\varphi$ is said to be of *bounded variation* ($BV$) if $\overline{\lim_T} \sum_{I \in D_t} \|\varphi(I)\| < +\infty$. The following results are well-known ([3*ab*, 2, 6*a*, 1*a*]).

$P_1$. *If $\varphi$ is q.a. on M, then it is BC-integrable over M, $M \subset A$.*

$P_2$. *If $\varphi$ is q.a. and BV on A then $\varphi$ and $\|\varphi\|$ are q.a. on M, for every $M \subset A$.*

Thus, any sets of conditions garanteeing that the function $\Phi$ is q.a. and $BV$ on $A$ is an existence theorem for the $W$-integral $BC\int_A F(p,\varphi)$. The following classical result is due to Cesari ([3*a*]) (see also [7*b*, 1*b*]).

THEOREM 1. *Suppose that F satisfies the following conditions*:

($F_1$) *F is bounded and uniformly continuous on $K \times S_1$, where $S_1 = \{x \in X: \|x\| = 1\}$;*

($F_2$) $F(k, tx) = tF(k, x)$, *for every* $t \geq 0$, $(k, x) \in K \times X$;

*suppose that the function p satisfies the condition*:

($\gamma$) *given* $\varepsilon > 0$ *there exists* $t_1 = t_1(\varepsilon)$ *such that for every* $t_0 \gg t_1$ *there exists* $t_2 = t_2(\varepsilon, t_0)$ *such that, if* $t \gg t_2$ *then* $\max_I \max_{J \subset I} d(p(J), p(I)) < \varepsilon$, *where* $D_{t_0} = [I]$, $D_t = [J]$;

*and that the function $\varphi$ is q.a. and BV on A.*

*Then the function $\Phi$ is q.a. and BV on A.*

Note that condition ($\gamma$) is a continuity assumption on the function $p$. For this reason Theorem 1 found many applications to the case of continuous $BV$ varieties (see [6] for a survey).

## 3. THE *BV* CASE.

In [1*d*] we proposed a joint condition on the couple of set functions $(p, \varphi)$ that allows to improve the previous result. We refer to [1*d*] for all the proofs and the details.

DEFINITION 2. We say that the couple $(p, \varphi)$ is *$\Gamma$-quasi-additive* ($\Gamma$-q.a.) if ($\Gamma$-q.a.) *given* $\varepsilon > 0$, *there exists* $0 < \sigma = \sigma(\varepsilon) \leq \varepsilon$ *and* $t_1 = t_1(\varepsilon)$ *such that for every* $t_0 \gg t_1$ *there exists* $t_2 = t_2(\varepsilon, t_0)$ *such that if* $t \gg t_2$ *then*

$$\text{i)}\ \sum_I \left\| \sum_{J \in \Gamma_I} \varphi(J) - \varphi(I) \right\| < \varepsilon,$$

$$\text{ii)}\ \sum_I \left\| \sum_{J \notin \Gamma_I} s(J, I)\, \varphi(J) \right\| < \varepsilon,$$

$$\text{iii)}\ \sum_J \left[ 1 - \sum_I s(J, I) \right] \|\varphi(J)\| < \varepsilon,$$

*where* $D_{t_0} = [I]$, $D_t = [J]$ *and* $\Gamma_I$ *is a subfamily* (even empty) *of the set*

$$\{J \subset I:\ d(p(J), p(I)) < \sigma\}.$$

The following propositions point out the connections between new condition ($\Gamma$-q.a.) and the previous ones.

$P_3$. *If $\varphi$ is q.a. on A and p satisfies condition ($\gamma$), then the couple $(p, \varphi)$ is $\Gamma$-q.a.*

$P_4$. *If the couple* $(p, \varphi)$ *is* $\Gamma$*-q.a., then* $\varphi$ *is q.a. on* $A$.

$P_5$. *If* $(p, \varphi)$ *is* $\Gamma$*-q.a. and* $\varphi$ *is BV, then the couple* $(p, \|\varphi\|)$ *is* $\Gamma$*-q.a. with respect to the same* $\sigma$ *and* $\Gamma_I$.

However note that condition ($\Gamma$-q.a.) does not necessarily implies that $p$ satisfies condition ($\gamma$), as applications show.

Condition ($\Gamma$-q.a.) furnishes the following existence result which extends Theorem 1 to the case of *BV* varieties, possibly discontinuous, as the applications emphasize.

Theorem 3. *Suppose that the function F satisfies conditions* ($F_1$) *and* ($F_2$), *and that the couple* $(p, \varphi)$ *is* $\Gamma$*-q.a. and* $\varphi$ *is BV. Then the function* $\Phi$ *is q.a. and BV on* $A$.

Note that an analogous theorem can be proved for the non-parametric *W*-integral. Moreover, even in this new general situation, the *W*-integral admits a Lebesgue-Stieltjes integral representation, extending therefore Cesari's result in [3*b*]. For the non-parametric case a Tonelli-type result, comparing *W*-integral with a corresponding Lebesgue-Stieltjes integral, still holds. We refer to [1*de*] for the details.

## 4. The lower semicontinuity of *W*-integral.

In [7*a*] Warner proposed a first result on the lower semicontinuity of the parametric *W*-integral which contains, as applications, the classical theorems by Tonelli and Turner. Successively in [1*b*] we presented a modified version of such a result in order to widen the field of applications. Again with the same idea in mind, we proved in [1*f*] a new semicontinuity theorem for both the parametric and the non-parametric *W*-integral. For these semicontinuity theorems we adopt the same device of replacing $T(\omega(I)), \varphi(I)$ by a pair $(P(I), \varphi(I))$ of interval functions as in no. 2 for the existence of *W*-integrals. Moreover we again connect the assumptions on the two interval functions. In other words, we propose a global convergence condition on the sequence of couples $((p_n, \varphi_n))_{n\geqslant 0}$ which is the following one.

Definition 4. We say that the sequence $((p_n, \varphi_n))_n$ $\Delta$*-converge to* $(p_0, \varphi_0)$ *if* ($\Delta$) *given a subsequence* $((p_{k_n}, \varphi_{k_n}))_n$ *and fixed* $\varepsilon > 0$ *and* $t_1 \in T$, *then there exist* $t_0 = t_0((k_n)_n, \varepsilon, t_1) \gg t_1$ *and a subsequence* $(m_{k_n})_n$ *such that, for every* $n \in \mathbb{N}$ *there exists* $t_n = t_n(\varepsilon, t_1, n)$ *such that for every* $t \gg t_n$ *there exists* $t_* = t_*(\varepsilon, t_1, n, t) \gg t$ *with*

$$\sum_I \left\| \varphi_0(I) - \sum_{J\in\Delta_I} \varphi_{m_{k_n}}(J) \right\| < \varepsilon$$

*where* $D_{t_0} = [I]$, $D_t = [J]$ *and* $\Delta_I$ *is a subfamily (even empty) of the set*

$$\{J \subset I: d(P_0(I), p_{m_{k_n}}(J)) < \varepsilon\}.$$

Note that $\Delta$-convergence on $((p_n, \varphi_n))_{n\geqslant 0}$ is less restrictive that the previous conditions assumed on the sequences $(p_n)_{n\geqslant 0}$ and $(\varphi_n)_{n\geqslant 0}$ separately. Moreover it is much more suitable for our scope since it finds application to $L_1$-convergence of equi *BV* varieties.

The general lower semicontinuity result of [1*f*] is the following one.

THEOREM 5. *Suppose that the function F satisfies conditions* $(F_1)$, $(F_2)$ *and is seminormal and that* $((p_n, \varphi_n))_n$ *is a sequence* $\Delta$*-converging to* $((p_0, \varphi_0))$, *with* $((p_n, \varphi_n))$ $\Gamma$*-q.a. and* $\varphi_n$ *BV,* $n \geq 0$. *Then*

$$\varliminf_{n \to +\infty} BC \int_A F(p_n, \varphi_n) \geq BC \int_A F(p_0, \varphi_0) .$$

An analogous result holds for the non-parametric $W$-integral. As a consequence of both these general results, we obtain in [1*f*] some lower semicontinuity theorems for the weighted Weierstrass integral over *BV* curves and surfaces which extend the well-known results for length, for area, for the weighted generalized variation (see [6] for a survey).

## 5. THE $W$-INTEGRAL OVER A *BV* CURVE

In order to illustrate the existence result of Section 3, now we apply Theorem 3 to the particular case of the $W$-integral over a *BV* curve of the space $\mathbb{R}^n$. Again we refer to [1*d*] for all details and proofs.

Let $x: [a, b] \to \mathbb{R}^n$ be a *BV* curve and let $E_x$ denote the set of the points of essential continuity for $x$, i.e. $E_x = \{c \in [a, b]: x(c) = x(c-0) = x(c+0)\}$; as it is well-known $[a, b] - E_x$ is a null set.

Let $\{I\}$ be the family of all the closed sub-intervals of $[a, b]$ whose end-points belong to $E_x$ and let $\mathcal{D}_x$ be the collection of all the finite divisions of the type $D = [I_1, \ldots, I_n]$ with $I_i \in \{I\}$ and $\bigcup_{i=1}^{N} I_i = [a_1, a_{N+1}]$.

We consider the mesh function $\delta: \mathcal{D}_x \to R$ defined by $\delta(D) = \max\{(a_1 - a), (b - a_{N+1}), |I|, I \in D\}$ which makes $\mathcal{D}_x$ a directed set.

Observe now that for every $I \in \{I\}$, $\max_{t \in I} \|\Delta x(t)\| = m_I$ exists, where $\Delta x(t) = x(t+0) - x(t-0)$, and we denote by $t_I \in I$ a point such that $\|\Delta x(t_I)\| = m_I$.

Let $P_x: \{I\} \to R^n$ be an interval function such that $P_x(I) \in \operatorname{cl}\operatorname{co} x(I)$, and consider the function $\Delta x: \{I\} \to \mathbb{R}^n$ defined by $\Delta x(I) = \Delta x([\alpha, \beta]) = x(\beta) - x(\alpha)$.

The following condition on the function $P_x$ will play a fundamental role in the existence result.

DEFINITION 6. We say that $P_x$ satisfies *condition* $(\gamma')$ if

$(\gamma)'$ *for every* $\varepsilon > 0$ *there exists* $0 < \sigma = \sigma(\varepsilon) \leq \varepsilon$ *and* $\eta = \eta(\varepsilon) > 0$ *such that for every* $D_0 = [I] \in \mathcal{D}_x$ *with* $\delta(D_0) < \eta$ *there exists* $\lambda = \lambda(\varepsilon, D_0) > 0$ *in such a way that, if* $D = [J] \in \mathcal{D}_x$ *with* $\delta(D) < \lambda$, *then for every* $I \in D_0$, *there exists* $J_I \in D$ *with* $J_I \subset I$, $t_I \in J_I$ *and* $\|P_x(I) - P_x(J_I)\| < \sigma$.

LEMMA 7. *If* $P_x$ *satisfies condition* $(\gamma)'$ *then the couple* $(P_x, \Delta x)$ *is* $\Gamma$*-q.a. with respect to* $\mathcal{D}_x$ *and* $\delta$.

Let us consider now the function $p_x = (p_x^1, \ldots, p_x^n)$ defined by $p_x^i(I) = \lambda^i \cdot \operatorname{inf\,ess}(x^i, I) + (1 - \lambda^i) \operatorname{sup\,ess}(x^i, I)$, $I \in \{I\}$, where $0 \leq \lambda^i \leq 1$ is fixed, $i = 1, \ldots, n$. Observe that the function $p_x$ satisfies condition $(\gamma)'$, thus the following result can be proved.

THEOREM 8. *Let $F: K \times \mathbb{R}^n \to \mathbb{R}$, $K \subset \mathbb{R}^n$, be a function satisfying conditions $(F_1)$ and $(F_2)$ and let $x: [a,b] \to K$ be a BV function. Then the interval function $\Phi: \{I\} \to \mathbb{R}$ defined by $\Phi(I) = F(p_x(I), \Delta x(I))$ is q.a. and BV with respect to $\mathcal{D}_x$ and $\delta$.*

*Moreover the following integral representation holds*

$$BC \int_{[a,b]} \Phi = \int_a^b F(\pi(t), d\mu/d\|\mu\|(t))\, d\|\mu\|(t),$$

*where* $\pi^i(t) = \lambda^i \min(x^i(t+0), x^i(t-0)) + (1-\lambda^i)\max(x^i(t+0), x^i(t-0))$, $i = 1, \ldots, n$, *and $\mu$ is the variation measure associated to $x$. In particular, if $x$ is absolutely continuous in the generalized sense, we have*

$$BC \int_{[a,b]} \Phi = \int_a^b F(x(t), x'(t))dt.$$

Theorem 8 allows to define $W$-integral over a $BV$ curve obtaining an extension of the well-known results for continuous $BV$ curve (see [6] for a survey). Moreover note that, since the function $p_x$ does not satisfy condition $(\gamma)$ of Theorem 1, the above result could not be proved as a cosequence of the already known results for the $W$-integral.

Now we wish to point out the operativity of Burkill-Cesari algorithm by calculating the value of the $W$-integral $BC \int_{[a,b]} \Phi$ in the following example.

EXAMPLE 9. Let $(a_n)_n$ be a sequence in $[0,1]$ decreasing to 0 with $a_1 = 1$, and let $(c_n)_n$ be a decreasing sequence of positive number such that $\sum_{n=1}^{\infty} (c_n - c_{n+1}) < +\infty$. We consider the $BV$ curve $x: [0,1] \to \mathbb{R}$ defined by $x(t) = c_n$ for $t \in ]a_{n+1}, a_n]$, $n \in \mathbb{N}$, and $x(0) = c_1$. Then $E_x = [0,1] \setminus (\{a_n, n \in \mathbb{N}\} \cup \{0\})$.

Let $F: [0, c_1] \times \mathbb{R} \to \mathbb{R}$ be defined by $F(p,q) = p|q|$ and let consider the interval function $\Phi_\alpha(I) = F(p_x(I), \Delta x(I))$, $I \in \{I\}$, where $p_x(I) = \alpha \inf\operatorname{ess}(x, I) + (1-\alpha) \cdot \sup\operatorname{ess}(x, I)$, $0 \leq \alpha \leq 1$.

Note that, if $I \cap \{a_n, n \in \mathbb{N}\} \neq \emptyset$, then $\Phi_\alpha(I) = 0$ and if $I \cap \{a_n, n \in \mathbb{N}\} = \{a_{\bar n}\}$, then $\Phi_\alpha(I) = (\alpha c_{\bar n} + (1-\alpha) c_{\bar n - 1})(c_{\bar n - 1} - c_{\bar n})$. Therefore it is easy to see that

$$BC \int_{[0,1]} \Phi_\alpha = \lim_{m \to +\infty} \sum_{n=2}^{m} (\alpha c_n + (1-\alpha) c_{n-1})(c_{n-1} - c_n) = \sum_{n=2}^{\infty} (\alpha c_n + (1-\alpha) c_{n-1})(c_{n-1} - c_n).$$

Let now consider the sequence of polygonals $x_n: [0,1] \to \mathbb{R}$ defined by $x_n(t) = c_n$, for $t \in [0, a_n]$, $x_n(t) = c_m$, for $t \in ]a_{m+1} + (a_n - a_{n+1}), a_m]$, $m = 1, \ldots, n-1$, and $x_n$ is linear elsewhere. Then $(x_n)_n$ converges to $x$ pointwise on $]0,1]$ and moreover, denoted by $BC \int \Phi_{n,\alpha}$ the $W$-integral relative to $x_n$, we have that

$$BC \int \Phi_{n,\alpha} = \int_0^1 x_n(t)\, |x_n'(t)|\, dt = \sum_{m=1}^{n-1} \int_{a_{m+1}}^{(a_{m+1} + a_n - a_{n+1})} \left[ \frac{c_m - c_{m+1}}{a_n - a_{n+1}} (t - a_{m+1}) + c_{m+1} \right] \cdot$$

$$\cdot \frac{c_m - c_{m+1}}{a_n - a_{n+1}}\, dt = \sum_{m=1}^{n-1} \frac{c_m + c_{m+1}}{2} (c_m - c_{m+1}).$$

Therefore we have that

$$\lim_{n\to+\infty} BC\int \Phi_{n,\alpha} = \sum_{n=2}^{\infty} \frac{c_n + c_{n-1}}{2}(c_{n-1} - c_n) \geq BC\int \Phi_\alpha \qquad \text{iff } \tfrac{1}{2} \leq \alpha \leq 1 .$$

Moreover for $\alpha = 1/2$ we get the following approximation result

$$BC\int \Phi_{1/2} = \lim_{n\to+\infty} BC\int \Phi_{n,1/2} .$$

Finally, we denote by $S(x) = \inf_{\{x_n\}} \varliminf_{n\to+\infty} \int_0^1 x_n(t)\,|x_n'(t)|\,dt$ the Serrin-type functional [5], where the least upper bound is taken with respect to any sequence of $AC$ curves $(x_n)_n$ converging to $x$ pointwise a.e. on $[0,1]$. Then it can be proved that

$$S(x) = BC\int \Phi_{1/2} .$$

In other words, the $W$-integral and the corresponding Serrin functional coincide, if we choose $P_x(I) = 1/2\ (\operatorname{inf\,ess}(x,I) + \operatorname{sup\,ess}(x,I)) \in \operatorname{cl}\operatorname{co} x(I)$.

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