Alessandro Caterino, Maria Cristina Vipera

Wallman-type compaerifications and function lattices


Accademia Nazionale dei Lincei

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Topologia. — Wallman-type compactifications and function lattices (*). Nota di Alessandro Caterino e Maria Cristina Vipera (**), presentata (***) dal Socio G. Zappa.

ABSTRACT. — Let \( F \subset C^*(X) \) be a vector sublattice over \( \mathbb{R} \) which separates points from closed sets of \( X \). The compactification \( e_F X \) obtained by embedding \( X \) in a real cube via the diagonal map, is different, in general, from the Wallman compactification \( \omega(Z(F)) \). In this paper, it is shown that there exists a lattice \( F_z \) containing \( F \) such that \( \omega(Z(F)) = e_F X \). In particular this implies that \( \omega(Z(F)) \supseteq e_F X \). Conditions in order to be \( \omega(Z(F)) = e_F X \) are given. Finally we prove that, if \( aX \) is a compactification of \( X \) such that \( Cl_{aX}(aX\setminus X) \) is 0-dimensional, then there is an algebra \( A \subset C^*(X) \) such that \( \omega(Z(A)) = e_A X = aX \).

Key words: Compactifications; Normal bases; Function lattices; Zero-sets.

RIASSUNTO. — Compattificazione di tipo Wallman e reticoli di funzioni. — Sia \( F \subset C^*(X) \) reticolo ed \( \mathbb{R} \)-spazio vettoriale che separa i punti dai chiusi. La compattificazione \( e_F X \), ottenuta immergendo \( X \) in un cubo reale mediante l’applicazione diagonale \( e_F \), è in generale diversa dalla compattificazione di Wallman \( \omega(Z(F)) \). In questa nota si dimostra che esiste un reticolo \( F_z \) contenente \( F \) tale che \( \omega(Z(F)) = \omega(Z(F_z)) = e_F X \). Ciò implica in particolare che \( \omega(Z(F)) \supseteq e_F X \). Si danno condizioni necessarie e sufficienti affinché valga l’uguaglianza. Infine si dimostra che, se \( aX \) è una compattificazione di \( X \) tale che \( Cl_{aX}(aX\setminus X) \) è zero-dimensionale, allora esiste un’algebra \( A \) di funzioni continue definite su \( X \) tale che \( \omega(Z(A)) = e_A X = aX \).

1. INTRODUCTION

Let \( X \) be a Tychonoff space and let \( F \) be a subset of \( C^*(X) \), the ring of all bounded continuous real functions on \( X \).

If the diagonal map \( e_F : X \to \mathbb{R}^F \) is an embedding, in particular if \( F \) separates points from closed sets, then we denote by \( e_F X \) the compactification \( \overline{e_F(X)} \) of \( X \).

One has another compactification naturally associated with \( F \), namely \( e_F X \). This compactification \( F \) is a normal base: the Wallman-type compactification \( \omega(Z(F)) \) (see [10]). These two compactifications can be very different: for instance it is known that, if \( X \) is a metrizable non-compact locally compact space and \( F = \{ f \in C^*(X) \mid \lim_{x \to \infty} f(x) = \gamma \in \mathbb{R} \} \) then \( Z(F) \) is a normal base and \( \omega(Z(F)) = \beta X \supseteq e_F X \).

More generally, if \( aX \) is a \( T_2 \)-compactification of \( X \), and \( C_a \) is the ring of the real continuous functions which extend to \( aX \), then \( Z(C_a) \) is a normal base and \( \omega(Z(C_a)) \supseteq aX = e_{C_a} X \) (see [7]). In this paper it is shown that, for every compactification \( \omega(Z(F)) \), where \( F \subset C^*(X) \) satisfies suitable conditions to guarantee that \( Z(F) \) is a normal base, there exists a set \( G \subset C^*(X) \) such that \( \omega(Z(F)) = \omega(Z(G)) = e_G X \).

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(**) Indirizzo degli Autori: Dipartimento di Matematica dell’Università - Via Vanvitelli 1 - 06100 Perugia. Tel. 075/40242.

More precisely, if $F$ is a vector sublattice of $C^*(X)$ such that $R \subset F$ and $e_F$ is an embedding, then we remark that $Z(F)$ is a normal base and we show how to enlarge $F$ to a lattice $F_1$ such that $Z(F_1) = Z(F)$ and $\omega(Z(F_1)) = e_F X$. (This implies in particular $\omega(Z(F)) \supseteq e_F X$).

The above result allows us to get some equivalent conditions for $\omega(Z(F))$ to be equal to $e_F X$.

Finally, we prove that, if $\alpha X$ is a compactification such that $\text{Cl}_{\alpha X}(\alpha X \setminus X)$ is 0-dimensional, then there exists a subring $A$ of $C_\alpha$ such that $\alpha X = e_A X = \omega(Z(A))$.

2. Preliminary results

All spaces considered are Tychonoff.

We recall that, for a given (Hausdorff) compactification $\alpha X$ of $X$, the map $f \mapsto f^*$ (where $f^*$ is the extension of $f$ to $\alpha X$) is an algebra-isomorphism and a lattice-isomorphism between $C_\alpha$ and $C(\alpha X)$ which is also a homeomorphism with respect to the uniform convergence topology (u.c. topology, for short).

Following [3], we say that $F \subset C^*(X)$ generates a compactification $\alpha X$ if $\alpha X$ is equivalent to $e_F X$ (in the usual sense). In this case, we have $F \subset C_\alpha$. If $F \subset G \subset C^*(X)$ and $F, G$ generate $\alpha_1 X, \alpha_2 X$ respectively, then $\alpha_1 X \leq \alpha_2 X$.

From now on, we do not distinguish between equivalent compactifications.

We wish to recall a result about normal bases, which will be used later. (See [10], for instance, for the definitions of normal bases and Wallman-type compactifications).

Let $\alpha X$ be a compactification of $X$, $\mathcal{C}$ a family of closed subsets of $X$. Then we put

$$\mathcal{C} = \{\text{Cl}_{\alpha X}(S) | S \in \mathcal{C}\}.$$

**Proposition 1.** Let $\alpha X$ be a compactification of $X$ and let $\mathcal{C}$ be a lattice of closed subsets of $X$. Then the following conditions are equivalent:

(i) $\mathcal{C}$ is a normal base for $X$ and $\alpha X = \omega(\mathcal{C})$;

(ii) $\mathcal{C}$ is a base for the closed subsets of $\alpha X$, and disjoint elements of $\mathcal{C}$ have disjoint closures in $\alpha X$.

This proposition is a slight modification of Theorem 1 in [4] (there ascribed to Shanin). Its condition (b) (that is $A \cap B = \overline{A \cap B}$ for all $A, B \in \mathcal{C}$) can be replaced by the requirement that disjoint elements of $\mathcal{C}$ have disjoint closures in $\alpha X$, in view of Lemma 2.3 in [5].

3. Normal bases of zero-sets

Let $X$ be any (Tychonoff) space. Then one has:

**Proposition 2.** Let $F \subset C^*(X)$ be a lattice such that $R + F \subset F$, $\mathbb{R}F \subset F$ and which separates points from closed sets. Suppose that, $\forall f, g \in F$ such that $Z(f) \cap Z(g) = \emptyset$, the following conditions are equivalent:

(i) $\omega(Z(F)) \supseteq e_F X$;

(ii) $\mathcal{C}$ is a normal base for $X$ and $\alpha X = \omega(\mathcal{C})$;

(iii) $\mathcal{C}$ is a base for the closed subsets of $\alpha X$, and disjoint elements of $\mathcal{C}$ have disjoint closures in $\alpha X$. 

This proposition is a slight modification of Theorem 1 in [4] (there ascribed to Shanin). Its condition (b) (that is $A \cap B = \overline{A \cap B}$ for all $A, B \in \mathcal{C}$) can be replaced by the requirement that disjoint elements of $\mathcal{C}$ have disjoint closures in $\alpha X$, in view of Lemma 2.3 in [5].
we have $\text{Cl}_{eX}(Z(f)) \cap \text{Cl}_{eX}(Z(g)) = \emptyset$. Then $Z(F)$ is a normal base and $\omega(Z(F)) = \varepsilon_F X$.

**Proof.** Since $Z(F)$ is a lattice, in view of proposition 1, we only have to prove that 
\[ \{\text{Cl}_{eX}(Z(f)) \mid f \in F\} \] is a base for the closed subsets of $\alpha X = e_F X$. By [6], prop. 1, we have that $F$ is a dense subset of $C_a$ with respect to the u.c. topology. Hence $F^* = \{f^* \mid f \in F\}$, is dense in $C(\alpha X)$, therefore $F^*$ separates points from closed subsets of $\alpha X$.

Now, let $A$ be a closed subset of $\alpha X$, $y \in \alpha X \setminus A$ and let $f \in F^*$ be such that $f^*(y) \neq f^*(A)$. If $a = f^*(y)$, we set $g = |f - a|$, then $g \in F$, $g^*(y) = 0$ and, for some $b > 0$, $g^*(z) \geq b$ $\forall z \in A$. We note that $Z = g^{-1}([b/2, +\infty[) = Z((g - b/2) \wedge 0) \in Z(F)$ and $y \notin (g^*)^{-1}([b/2, +\infty[) \cap \text{Cl}_{eX}(Z)$. It remains only to show that $A \subset \text{Cl}_{eX}(Z)$. Let $t \in A$ and let $U$ be a neighbourhood of $t$ in $\alpha X$. Since also $(g^*)^{-1}([b/2, +\infty[)$ is a neighbourhood of $t$ in $\alpha X$ and $X$ is dense in $\alpha X$, we have $\emptyset = X \cap (U \cap (g^*)^{-1}([b/2, +\infty[)) = U \cap Z$. Hence $t \in \text{Cl}_{eX}(Z)$. $lacksquare$

In the above proposition the hypothesis «$F$ separates points from closed sets» can be replaced by «$e_F$ is an embedding». In fact, these two conditions are equivalent when $F$ is a lattice such that $RF \subset F$ and $R + F \subset F$.

If $f \in C^* (X)$, we set 
\[ S(f) = \{f^{-1}([a, b]) \mid a, b \in R\} \]
and, if $F \subset C^* (X)$ we put $S(F) = \bigcup_{f \in F} S(f)$.

We note that $S(F) = Z(F)$ when $F$ is a lattice such that $R + F \subset F$, $RF \subset F$. In fact 
\[ f^{-1}([a, b]) = f^{-1}([a, +\infty[ \cap f^{-1}([-\infty, b]) = Z((f - a) \wedge 0) \cap Z((f - b) \vee 0) \in Z(F). \]
Under the same hypotheses on $F$, if we put 
\[ F_z = \{g \in C^* (X) \mid S(g) \subset Z(F)\}, \]
then we have $F \subset F_z$ and $Z(F_z) = Z(F)$. Furthermore one can easily prove that $F_z$ is a lattice and $R + F_z \subset F_z$, $RF_z \subset F_z$.

**Theorem 3.** Let $F \subset C^* (X)$ be a vector sublattice over $R$, which contains all constant functions and separates points from closed sets. Then $Z(F)$ is a normal base and $\omega(Z(F)) = \omega(Z(F_z)) = \varepsilon_{F_z} X$. Therefore $\omega(Z(F)) \geq \varepsilon_F X$.

**Proof.** Put $\alpha X = e_{F_z} X$. Let $f, g \in F$ be such that $Z(f) \cap Z(g) = \emptyset$. If 
\[ h = \frac{|f|}{|f| + |g|}, \]
once one easily sees that $h \in F_z$. Then $h$ has a continuous extension $h^*$ to $\alpha X$, hence $(h^*)^{-1}(0)$, $(h^*)^{-1}(1)$ are disjoint closed subsets of $\alpha X$ containing respectively $Z(f)$, $Z(g)$. Since $Z(F_z) = Z(F)$, then $F_z$ satisfies all the hypotheses of Proposition 2. $lacksquare$
PROPOSITION 4. Let $F \subseteq C^*(X)$ be a vector sublattice over $\mathbb{R}$ which contains all constant functions and separates points from closed sets. Then the following are equivalent:

(i) $\omega(Z(F)) = e_F X$;

(ii) For every $f, g \in F$, $Z(f) \cap Z(g) = \emptyset$ implies

\[ \text{Cl}_{aX}(Z(f)) \cap \text{Cl}_{aX}(Z(g)) = \emptyset; \]

(iii) $F_z \subseteq C_{co}$;

(iv) $F$ is a dense subset of $F_z$ with respect to u.c. topology.

PROOF. (i) $\Rightarrow$ (ii) is a consequence of well known facts about normal bases (see also prop. 1).

(ii) $\Rightarrow$ (iii). If $b \in F_z$ and $a > b$, then $b^{-1}([a, +\infty[), b^{-1}([-\infty, b))$ are disjoint sets belonging to $Z(F)$. So they have disjoint closures in $aX$. Then, by [6], cor. 3, $b$ has a continuous extension to $e_F X$.

(iii) $\Rightarrow$ (i) The hypothesis implies $e_F X = e_F X$, which is equal to $\omega(Z(F))$ by Thm. 3.

Finally, since $F$ is dense in $C_a$ and $C_a$ is a closed set in $C^*(X)$, we have (iii) $\Rightarrow$ (iv). $lacksquare$

The following proposition establishes that, for every compactification $aX$ whose remainder is sufficiently disconnected (in a sense to be made precise), there exists a subring $A$ of $C_a$ such that $aX = e_A X = \omega(Z(A))$.

PROPOSITION 5. Let $aX$ be a compactification of $X$ and let $A = \{ f \in C_a \mid \forall p \in \text{Cl}_{aX}(aX \setminus X) \text{ there is a neighbourhood } U \text{ of } p \text{ such that } f^*|U \text{ is constant} \}$. Then $aX = e_A X = \omega(Z(A))$ if and only if $\text{Cl}_{aX}(aX \setminus X)$ is 0-dimensional.

PROOF. First we show that disjoint elements of $Z(A)$ have disjoint closures in $aX$. Suppose that $Z(f) \cap Z(g) = \emptyset$ with $f, g \in A$.

Now, if $p \in \text{Cl}_{aX}(Z(f)) \cap \text{Cl}_{aX}(Z(g))$, let $U_f$ and $U_g$ be neighbourhoods of $p$ such that $f^*|U_f$ and $g^*|U_g$ are constant.

Then $f(x) = g(x) = 0$ for every $x \in U_f \cap U_g \cap X$, which is non-empty. This is a contradiction, because $Z(f) \cap Z(g) = \emptyset$. Now, let $Y = \text{Cl}_{aX}(aX \setminus X)$ be 0-dimensional. Since $A$ is obviously a lattice and an $R$-algebra, if we prove that $A$ generates $aX$ then, by applying Proposition 4, we obtain $aX = e_A X = \omega(Z(A))$. Therefore it is sufficient to prove that $A^*$ separates points of $aX$ (see [3], thm. 2.3).

Let $x, y \in aX$, $x \neq y$. First suppose that one of them, say $x$, does not belong to $Y$. Then choose a closed neighbourhood $V$ of $Y \cup \{y\}$ not containing $x$. If $b \colon aX \to \mathbb{R}$ is a continuous map such that $b(x) = 0$ and $b(V) = 1$, one has $b|X \in A$ and $b = (b|X)^*$ separates $x$ from $y$.

Now, suppose $x, y \in Y$. Since $Y$ is 0-dimensional, there exist disjoint closed subsets $C_1$ and $C_2$ of $Y$ (hence closed in $aX$) such that $x \in C_1, y \in C_2$ and $C_1 \cup C_2 = Y$. Let
$V_1$, $V_2$ be disjoint closed neighbourhoods in $\alpha X$ of $C_1$ and $C_2$ respectively, and let $k: \alpha X \rightarrow \mathbb{R}$ be a continuous map such that $k(C_1) = 0$ and $k(C_2) = 1$. As before $k|X \in A$ and $k(x) \neq k(y)$.

Conversely, suppose that $Y = \text{Cl}_{\alpha X}(\alpha X \setminus X)$ is not 0-dimensional. Then there is a connected subset $C$ of $Y$ which is not a singleton. Since locally constant functions defined on a connected space are constant, then $A^*$ does not separate points of $C$.

Moreover, it cannot happen that $\alpha X = \omega(Z(A))$. Indeed, since $A^*$ does not separate points of $\alpha X$, then $Z(A)$ is not a base for closed subsets of $\alpha X$, because $\text{Cl}_{\alpha X}(Z(f)) = Z(f^*)$ for all $f \in A$.

As a final remark, we point out that Brooks, in [7], used similar arguments to prove that, in the case of $X$ locally compact, $\alpha X = \omega(Z(A))$ if and only if $\alpha X \setminus X$ is 0-dimensional.

REFERENCES