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Calcolo delle variazioni. — *Existence of discontinuous absolute minima for certain multiple integrals without growth properties.* Nota (*) del Socio Straniero LAMBERTO CESARI.

ABSTRACT. — In the present paper the author discusses certain multiple integrals $I(s)$ of the calculus of variations satisfying convexity conditions, and no growth property, and the corresponding Serrin integrals $\mathfrak{I}(u)$, to which the existence theorems in [3, 4, 5] do not apply. However, in the present paper, the integrals $I(u)$ and $\mathfrak{I}(u)$ are reduced to simpler form $H(v)$ and $\mathcal{H}(v)$ to which the existence theorems above apply. Thus, we derive that $I(u) \leq \mathfrak{I}(u)$, $H(v) \leq \mathcal{H}(v)$, we obtain the existence of the absolute minimum for the Serrin forms $\mathfrak{I}(u)$ and $\mathcal{H}(v)$, and such minimum is given by BV functions, possibly discontinuous and not of Sobolev.

KEY WORDS: BV function; Property (Q); Property (F); Serrin integral.

RIASSUNTO. — *Esistenza di minimi assoluti discontinui per certi integrali multipli senza proprietà di crescita.* Nel presente lavoro si discutono certi integrali multipli $I(u)$ del calcolo delle variazioni, soddisfacenti condizioni di convessità, non aventi proprietà di crescita, e i corrispondenti integrali di Serrin $\mathfrak{I}(u)$, a cui i teoremi di esistenza in [3, 4, 5] non si applicano. Tuttavia, nel presente lavoro gli integrali $I(u)$ e $\mathfrak{I}(u)$ sono ridotti a forme più semplici $H(u)$ e $\mathcal{H}(v)$ a cui i teoremi di esistenza detti sopra sono applicabili. Così ne risulta che $I(u) \leq \mathfrak{I}(u)$, $H(v) \leq \mathcal{H}(v)$, e otteniamo l'esistenza del minimo assoluto per le forme di Serrin $\mathfrak{I}(u)$ e $\mathcal{H}(v)$, dato da funzioni BV, possibilmente discontinue e non di Sobolev.

INTRODUCTION

In 1936 Cesari [1] defined the concept of functions $f(t)$, $t \in G \subset \mathbb{R}^\nu$, of class $L_1(G)$ and of bounded variation (BV). We say that $f(t) = f(t_1, \ldots, t_\nu)$ is BV in an open interval G of $\mathbb{R}^\nu$, $\nu \geqq 1$, if $f \in L_1(G)$ and there exists a set of measure zero $E \subset G$ such that the total variations with respect to each variable t_j are functions of class L_1 of the remaining variables-total variations all computed by completely disregarding the values taken by f on the points of E. We shall state below the corresponding definition of BV functions in a bounded open subset G of $\mathbb{R}^\nu$. Recently Cesari, Brandi and Salvadori [3, 4, 5] proved existence theorems of the calculus of variations for functions $u(t)$, $t \in G \subset \mathbb{R}^\nu$, $\nu \geq 1$, of class BV, thus possibly discontinuous and not of Sobolev. Namely for a general integral

$$I(u) = \int_G f_0(t, u, Du)\,dt$$

of the calculus of variations, Cesari, Brandi and Salvadori considered the corrisponding integral of Serrin $\mathfrak{I}(u)$, and proved existence theorems for the absolute minimum of the integral $\mathfrak{I}(u)$ in classes Ω of functions u of equibounded total variations, and in such situation they proved that $I(u) \leq \mathfrak{I}(u)$.

In the present paper we discuss certain multiple integrals $I(u)$ (2.1) satisfying convexity conditions, but no growth property, and their corresponding Serrin integrals

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$\mathfrak{I}(u)$. In the present paper the integrals $I(u)$ and $\mathfrak{I}(u)$ are reduced to simpler forms $H(v)$ and $\mathcal{H}(v)$, to which the general existence theorem applies which was proved by Cesari, Brandi and Salvadori in [5]. Thus, $I(u) \leqq \mathfrak{I}(u)$, $H(v) \leqq \mathcal{H}(v)$, and we obtain the existence of BV discontinuous absolute minima for the Serrin forms.

In §1 we recall definitions and a general existence theorem from [5], and in §2 we discuss the integrals of the present paper.

1. The statement of an existence theorem of the calculus of variations

Let us consider first an integral of the form

$$I(u) = \int_G f_0(t, u, Du)dt, \qquad dt = dt_1 \dots dt_\nu, \qquad (t, u(t)) \in A, \qquad t \in G, \tag{1.1}$$

where G is a fixed bounded domain of the t-space $\mathbb{R}^\nu$, $t = (t_1, \dots, t_\nu)$, with boundary ∂G possessing the cone property at every point $t \in \partial G$, and where u denotes an m-vector function $u(t) = (u_1 \dots, u_m)$ of t in G. Let A denote a fixed subset of the tu-space $\mathbb{R}^{\nu+m}$ whose projection on the t-space contains G. For every $i = 1, \dots, m$ let $[j]_i$ denote a finite system $1 \leqslant j_1 < \dots < j_{N_i} \leqslant \nu$ of indices which may depend on i, and $[j]_i$ can be empty for some i. Then Du in (1.1) denotes the system of $N = N_1 + \dots + N_m$ first order partial derivatives $Du = (D^j u_i, j \in [j]_i, i = 1, \dots, m)$. For every $(t, u) \in A$ let $Q(t, u)$ denote a subset of the ξ-space $\mathbb{R}^N$, $\xi = (\xi_1, \dots, \xi_N)$, and let M denote the set of all (t, u, ξ) with $(t, u) \in A$, $\xi \in Q(t, u)$. Let $f_0(t, u, \xi)$ denote a fixed real valued function defined on M, and for every $(t, u) \in A$ let $\tilde{Q}(t, u)$ denote the augmented set

$$\tilde{Q}(t, u) = [(\tau, \xi) | \tau \geqslant f_0(t, u, \xi), \xi \in Q(t, u)] \subset \mathbb{R}^{1+N}.$$

As usual, we say that the sets $\tilde{Q}(t, u)$ possess property (Q) at some $(t_0, u_0) \in A$ provided

$$\tilde{Q}(t_0, u_0) = \bigcap_{\delta > 0} \operatorname{cl} \operatorname{co} \Big[\bigcup \tilde{Q}(t, u) | \, |t - t_0| + |u - u_0| \leqslant \delta, (t, u) \in A \Big].$$

We mention now the property $\tilde{F}_1'$ with respect to u, one of a number of variants of properties (F) which we have introduced in [5].

We say that the sets $\tilde{Q}(t, u)$, $(t, u) \in A$, have property $(\tilde{F}_1')$ with respect to u at the point $(t_0, u_0) \in A$ provided, given any number $\sigma > 0$ there are constants $C = C(t_0, u_0, \sigma) > 0$, $\delta(t_0, u_0, \sigma) > 0$ such that, for any set of measurable vector functions $u(t)$, $\eta(t)$, $\xi(t)$, $t \in L$, on any measurable set $L \subset G$ with $(t, u(t)) \in A$, $|u(t) - u_0| > \sigma$, $(\eta(t), \xi(t)) \in \tilde{Q}(t, u(t))$ for $t \in L$, $|t - t_0| \leqslant \delta$, other measurable vector functions $\bar{u}(t)$, $\bar{\eta}(t)$, $\bar{\xi}(t)$, $t \in L$, can be found such that

$$(t, \bar{u}(t)) \in A, \qquad |\bar{u}(t) - u_0| \leqslant \sigma, \qquad (\bar{\eta}(t), \bar{\xi}(t)) \in \tilde{Q}(t, \bar{u}(t)),$$
$$|\xi(t) - \bar{\xi}(t)| \leqslant C[|u(t) - \bar{u}(t)| + |t - t_0|],$$
$$\bar{\eta}(t) \leqslant \eta(t) + C[|u(t) - \bar{u}(t)| + |t - t_0|] \quad \textit{for } t \in L, \ |t - t_0| \leqslant \delta.$$

Conditions (Q) and (F) are sometimes called seminormality conditions (cfr. [2]). A weaker version of them, leading to some extensions of theorems A and B, is discussed in [6].

Let G be a bounded open subset of the t-space $\mathbb{R}^\nu$, $t = (t_1, \dots, t_\nu)$. For every $j =$

$=1,\dots,\nu$, let G_j' denote the projection of G on the t_j'-space $\mathbb{R}^{\nu-1}$, $t_j' = (t_1,\dots,t^{j-1},t^{j+1},\dots,t_\nu)=\tau$, and for any $\tau\in G_j'$ let r_τ denote the straight line $t_j'=\tau$. Then the intersection $G\cap r_\tau$ is the countable union of open intervals (α_s,β_s), or $G\cap r_\tau = \bigcup_s (\alpha_s,\beta_s)$. We say that a function $f\in L_1(G)$ is of bounded variation in the sense of Cesari in $G(BV)$ ([1], 1936) if there exists a set $E\subset G$ with $|E|=0$ such that, for every $j=1,\dots,\nu$, and for almost all $\tau\in G_j'$ the total variations $V_{js}=V(f(\cdot,\tau),(\alpha_s,\beta_s))$, computed disregarding the values taken by f on E, are finite, $V_j(\tau)=\sum_s V_{js}$ is finite, and $V_j(\cdot)\in L_1(G_j')$.

Cesari [1] also proved that f is BV in G if and only if the surface $S: z=f(t)$, $t\in G$, has finite generalized Lebesgue area $L(S)$; and thus the BV concept is independent of the direction of the axes in the t-space $\mathbb{R}^\nu$. Krickeberg ([8], 1957) proved that f is BV in G if and only if $f\in L_1(G)$ and the first order partial derivatives of f in the sense of distributions are finite measure μ_j, $j=l,\dots,\nu$, on G. Thus, a BV function in G has (generalized) partial derivatives $(D^j f)(t)$, $t\in G$, $j=l,\dots,\nu$, (cfr. [3]) which are functions of class $L_1(G)$, as well as derivatives μ_j, $j=l,\dots,\nu$, in the sense of distributions, which are finite measures on G. As proved by Cafiero and Fleming (cfr. [3]), any sequence $[f_k]$ of BV functions on G with equibounded total variations and equibounded mean values in G (say, $V(f_k)\leqslant V_0$, $|\text{m.v.}(f_k)|\leqslant M_0$), possesses a subsequence f_{k_s} which is strongly convergent in $L_1(G)$ toward a BV function f (with $V(f)\leqslant V_0$, $|\text{m.v.}(f)|\leqslant M_0$).

We shall consider the class Ω of all BV m-vector functions $u(t)=(u_1,\dots,u_m)$, $t\in G$, which are BV in G (*i. e.*, each component is BV), satisfying $(t,u(t))\in A$ *a. e.* in G; possibly satisfying Dirichlet boundary conditions, say $u(t)=w(t)$ on some part D of ∂G. We shall also restrict Ω to all those u whose total variations on G, say $V_0(f)$ or $V(f)$ (cfr. [3]), do not exceed a fixed constant W_0, and whose mean values in G do not exceed a fixed constant K, say $|\text{m.v.}(f)|\leqslant K$. For any $u\in\Omega$ let $\Gamma(u)$ denote the class of all sequences $[u_k]$ of elements $u_k\in\Omega$, $u_k\in W^{1,1}(G)$, $u_k\to u$ in $L_1(G)$. Then we take $\mathfrak{I}(u) = +\infty$ if $\Gamma(u)$ is empty, and otherwise we take

$$\mathfrak{I}(u)=\inf_{\Gamma(u)} \varliminf_{k\to\infty} I(u_k).$$

This the Serrin-type integral $\mathfrak{I}$ we have associated to $I(u)$ in [3, 4, 5]. If the total variations $V(u)$ for $u\in\Omega$ are equibounded, then $I(u)\leqslant\mathfrak{I}(u)$ as proved in [4]. Moreover, $\mathfrak{I}$ is an extension of I in the sense that $\mathfrak{I}(u)=I(u)$ for $u\in W^{1,1}(G)$ (cf. [5]).

THEOREM A (an existence theorem, cfr. [5]). Let G be a bounded domain of the t-space $\mathbb{R}^\nu$ whose boundary has the cone property, let A be a compact subset of the tu-space $\mathbb{R}^{\nu+m}$ whose projection on the t-space covers G, and for every $(t,u)\in A$ let $Q(t,u)$ denote a given closed convex subset of the ξ-space $\mathbb{R}^N$. Let M denote the set of all (t,u,ξ) with $(t,u)\in A$, $\xi\in Q(t,u)$, and assume that M is closed, that the real valued function $f_0(t,u,\xi)$ is lower semicontinuous on M, and that there is a function $\lambda(t)$, $t\in G$, $\lambda\in L_1(G)$, such that $f_0(t,u,\xi)\geqslant\lambda(t)$ for all $(t,u,\xi)\in M$.

Let us assume that the sets $\tilde{Q}(t,u)$ possess property (Q) with respect to (t,u) and property $(\tilde{F}_1')$ with respect to u at every point $(t_0,u_0)\in A$, with exception perhaps of a set of points whose t coordinate lies on a set H of measure zero in G. Let Ω denote a given closed class of BV vector functions $u(t)$, $t \in G$, whose total variations and mean values in G are equibounded, and for which at least one class $\Gamma(u)$ is not empty. Then the functional $\mathfrak{J}(u)$ has an absolute minimum in Ω.

2. A class of integrals of the calculus of variations without growth property

Let G be a bounded domain of the t-space $\mathbb{R}^\nu$, $t = (t_1, \dots, t_\nu)$, whose boundary ∂G has the cone property at every point. Let $u(t) = (u_1, \dots, u_m)$, $t \in G$, denote an $L_1(G)$ function, possibly discontinuous, of class BV. Let A be a compact subset of the tu-space $\mathbb{R}^{\nu+m}$ whose projection on the t-space covers G. Let $u(t) = w(t)$, $t \in D \subset \partial G$, denote a system of Dirichlet data on some components of u on some part D of ∂G. We consider here multiple integrals of the calculus of variations with constraints

$$(2.1)\qquad \begin{cases} I(u) = \displaystyle\int_G \sum_{i=1}^{m} \left| \sum_{j \in [j]_i} (F_{ij}(t,u))_{t_j} + F_i(t,u) \right| dt, & dt = dt_1 \dots dt_\nu, \\ (t, u(t)) \in A, & t \in G, \\ u(t) = w(t), & t \in D \subset \partial G, \end{cases}$$

where the F_{ij} are functions of class C^1 on A, and the F_i are Lipschitzian functions on A, hence $|F_{ij}(t,u) - F_{ij}(\bar{t},\bar{u})|$, $|F_i(t,u) - F_i(\bar{t},\bar{u})| \leq C[|u - \bar{u}| + |t - \bar{t}|]$ for (t,u), $(\bar{t},\bar{u}) \in A$, some constant C and all $i = 1, \dots, m$, $j \in [j]_i$. In (2.1) for every $i = 1, \dots, m$, we denote by $[j]_i$ a given system of integers $1 \leq j_1 < \dots < j_{N_i} \leq \nu$, and we denote by N the total number of such indices, or $N = N_1 + \dots + N_m$. Let $F(t,u) = [F_{ij}(t,u), j \in [j]_i, i = 1, \dots, m]$. We shall denote by Ω the class of all vector functions $u(t) = (u_1, \dots, u_m)$, $t \in G$, with $(t, u(t)) \in A$, $u \in BV(G)$, $V(u) \leq W_0$ for some constant W_0. We need not require explicitely that mean values of the functions u in G are equibounded. Instead, we require that D and G are so related that the boundedness of w in D and the equiboundedness of the total variations of the function u in G with $u(t) = w(t)$ in D, implies that the mean values of the functions u in G are equibounded. In Section 3 we shall see a situation in which this occurs naturally.

We consider now the transformation Φ which transforms any m-vector function $u(t) = (u_1, \dots, u_m)$, $t \in G$, of the class Ω, hence $(t, u(t)) \in A$, $u \in BV(G)$, $V(u) \leq W_0$, into the $(N+m)$-vector function v defined by

$$v(t) = \Phi u = (v_0, v_{ij}, j \in [j]_i, i = 1, \dots, m) = (v_0, v^*),$$

$$v_0(t) = u(t), \qquad v_{ij}(t) = F_{ij}(t, u(t)), \qquad v^* = F(t, u(t)), \qquad t \in G.$$

Let A_0 denote the projection of A onto the t-space $\mathbb{R}^\nu$, hence $G \subset A_0$ and A_0 is compact. If $A(t)$ denotes the section of A, that is, $A(t) = [u | (t,u) \in A] \subset \mathbb{R}^m$, and for every $t \in A_0$ we take $B(t) = [(v_0, v^*), v_0 \in A(t) \subset \mathbb{R}^m, v^* = F(t,u), u \in A(t), t \in G, v^* \in \mathbb{R}^N]$, thus $B(t) \subset \mathbb{R}^{m+N}$. Finally,

$$A = [(t,u) | t \in A_0, u \in A(t)] \subset \mathbb{R}^{\nu+m}, \qquad B = [(t,v) | t \in A_0, v \in B(t)] \subset \mathbb{R}^{\nu+m+N},$$

$$(t, u(t)) \in A, \ \ (t, v(t)) \in B, \ \ t \in G.$$

Let $\Omega' = \Phi(\Omega)$. Now problem (2.1) is reduced to another problem with constraints:

$$(2.2)\quad \begin{cases} H(v) = \int_G \sum_{i=1}^{m} \left| \sum_{j\in[j]_i} (v_{ij})_{t_j} + F_i(t, v_0) \right| dt, & dt = dt_1 \dots dt_\nu, \\ (t, v(t)) \in B, & t \in G,\ v \in \Omega', \\ v_0(t) = w(t), \quad v^*(t) = F(t, w(t)), & t \in D \subset \partial G. \end{cases}$$

It we denote by Dv the N-vector of derivatives $Dv = \{(v_{ij})_{t_j},\ i = 1,\dots, m,\ j \in [j]_i\}$, then we can write the problem above as

$$H(v) = \int_G f_0(t, v, Dv)dt, \qquad dt = dt_1 \dots dt_\nu,\quad v \in \Omega',$$

$$f_0(t, v, Dv) = \sum_{i=1}^{m} \left| \sum_{i\in[j]_i} (v_{ij})_{t_j} + F_i(t, v_0) \right|,$$

$$(t, v(t)) \in B, \qquad t \in G,\quad v \in \Omega',$$

$$v_0(t) = w(t), \quad v^*(t) = F(t, w(t)), \qquad t \in D \subset G.$$

By writing

$$(\xi_1 \dots, \xi_{N_1}, \xi_{N_1+1}, \dots, \xi_{N_1+N_2}, \dots, \xi_{N-N_m} + 1, \dots, \xi_N)$$

in lieu of

$$Dv = [(v_{ij})_{t_j}, j \in [j]_i, i = 1, \dots, m],$$

the integrand f_0 becomes

$$f_0(t, v, \xi) = \sum_{i=1}^{m} \left| \sum_{j\in[j]_i} \xi_{N_1+\dots+N_{i-1}+j} + F_i(t, v_0) \right|,$$

$$t \in G,\quad v(t) \in B(t) \subset \mathbb{R}^{m+N},\quad \xi \in \mathbb{R}^N.$$

THEOREM B (an existence statement for the integral (2.1)).

Let G be a bounded domain of the t-space $\mathbb{R}^\nu$ whose boundary has the cone property, let A be a compact subset of the tu-space whose projection on the t-space covers G, let F_{ij}, $j \in [j]_i$, $i = 1,\dots, m$, be functions of class C^1 on A and let F_i, $i = 1,\dots, m$, be Lipschitzian functions on A. Let Ω denote the class of all BV functions $u(t) = (u_1, \dots, u_m)$, $t \in G$, with $(t, u(t)) \in A$ *a.e.* in G, BV in G (that is, whose components are all BV), satisfying some Dirichlet-type boundary condition $u(t) = w(t)$, $t \in D \subset \partial G$, and whose total variations are equibounded (say $V(u) \leq W_0$ for all $u \in \Omega$ and some constant W_0). Then the Serrin form $\mathfrak{I}(u)$ of the integral (2.1) has an absolute minimum u in Ω, and $0 \leq I(u) \leq \mathfrak{I}(u)$.

Proof of Theorem B. First we study the integral (2.1).

We assumed A compact in the tu-space $\mathbb{R}^{\nu+m}$ and the functions F_{kj}, F_k continuous on the set A. Hence, the set B, as continuous transformation of A, is also compact in the tv-space $\mathbb{R}^{\nu+N+m}$.

For integral (2.1) we had no constraints on the values of the derivatives of the functions

u, or $(Du)(t) \in Q(t,u) = \mathbb{R}^N$. Hence, no constraints on the derivatives Dv of the functions v, either, or $\xi(t) \in Q'(t,v) = \mathbb{R}^N$. The sets $\tilde{Q}'(t,v)$ are now of the form

$$\tilde{Q}'(t,v) = [(\eta,\xi)|\eta \geqslant f_0(t,v_0,\xi), \xi \in \mathbb{R}^N] =$$

$$= \left[(\eta,\xi)|\eta \geqslant \sum_{i=1}^{m} \left| \sum_{j\in[j]_i} \xi_{N_1+\ldots+N_{i-1}+j} + F_i(t,v_0)\right|\right] \subset \mathbb{R}^{N+1}.$$

Hence, the sets $\tilde{Q}'(t,v)$ depend only on t and on the sole components $v_0 = u$ of v. These sets $\tilde{Q}'(t,v)$ are obviously closed and convex and convex in the $\eta\xi$-space $\mathbb{R}^{N+1}$.

We have assumed all functions $F_j(t,u)$ continuous on the compact set A; hence the functions $F_j(t,v_0)$ are continuous on the compact set B. Given $\varepsilon > 0$ there is $\delta > 0$ such that, for any two points $(\bar{t},\bar{v}) \in B$, $(t,v) \in B$ with $|t-\bar{t}| + |v-\bar{v}| < \delta$ we have $|F_i(t,v_0) - F_i(\bar{t},\bar{v}_0)| < \varepsilon/m$. Then, for $(\eta,\xi) \in \tilde{Q}'(t,v)$, we have

$$\eta \geqslant \sum_{i=1}^{m} \left| \sum_{j\in[j]_i} \xi_{N_1+\ldots+N_{i-1}+j} + F_i(t,v_0)\right| \geqslant$$

$$\geqslant \sum_{i=1}^{m} \left| \sum_{j\in[j]_i} \xi_{N_1+\ldots+N_{i-1}+j} + F_i(t,\bar{v}_0)\right| - |F_i(t,v_0) - F_i(t,\bar{v}_0)|.$$

For

$$\bar{\eta}(t) = \sum_{i=1}^{m} \left| \sum_{j\in[j]_i} \xi_{N_1+\ldots+N_{i-j}+j} + F_i(t,\bar{v}_0)\right|,$$

we have

$$\eta \geqslant \bar{\eta} - m(\varepsilon/m) = \bar{\eta} - \varepsilon, \qquad (\bar{\eta},\xi) \in \tilde{Q}'(t,v).$$

By the arbitrary of ε we see that the sets $\tilde{Q}'(t,v)$ have property (Q) at (t,v). Also, given measurable functions $v(t)$, $\eta(t)$, $t \in L$, $|t-\bar{t}| < \delta$, L measurable, with $(t,v(t)) \in A$, $|v(t) - \bar{v}| \geqslant 0$, $(\eta(t),\xi(t)) \in \tilde{Q}'(t,v(t))$, then we take

$$\bar{v}(t) = \bar{v}, \qquad \bar{\xi}(t) = \xi(t), \qquad \bar{\eta}(t) = \sum_{i=1}^{m} \left| \sum_{j\in[j]_i} \xi_{N_1+\ldots+N_{i-1}+j} + F_i(\bar{t},\bar{v})\right|,$$

and as before we have

$$\eta(t) = \sum_{i=1}^{m} \left| \sum_{j\in[j]_i} \xi_{N_1+\ldots+N_{i-1}+j} + F_i(t,v_0(t))\right| \geqslant \sum_{i-1}^{m} \left| \sum_{j\in[j]_i} \xi_{N_1+\ldots+N_{i-1}+j} + F_i(\bar{t},\bar{v}_0)\right| -$$

$$-|F_i(t,v_0(t)) - F_i(\bar{t},\bar{v}_0)| \geqslant \bar{\eta}(t) - mC[|v(t)-\bar{v}| + |t-\bar{t}|],$$

where $t \in L$, $|t-\bar{t}| \leqslant \delta$ and C is a Lipschitz constant for the functions F_i. Thus, the sets $\tilde{Q}'(t,v)$ satisfy property $(\tilde{F}_1')$ [5, p. 108].

Note that M is here the set $M = [(t,v,\xi)|(t,v) \in B, \xi \in \mathbb{R}^N]$ which is certainly a closed subset of $\mathbb{R}^{m+2N}$, and that f_0 is nonnegative, or $f_0(t,v_0,\xi) \geqslant \lambda(t) \equiv 0$ on M. Recall that Ω is the class of all m-vector functions $u(t) = (u_1,\ldots,u_m)$, $t \in G$, with $(t,u(t)) \in A$ *a.e.* in G, and BV in G (*i.e.*, with components which are all BV), equibounded total variations,

$$V(u) = \sum_i V(u_i) \leqslant W_0, \qquad u(t) = w(t) \qquad \text{on } D \subset \partial G.$$

Let Ω' denote the class of all vector functions $v = \Phi u$, $u \in \Omega$, or

$$\Omega' = \{v(t) = (v_0, v^*), v_0(t) = u(t), v^*(t) = F(t, u(t)), t \in G, u \in \Omega\},$$

hence

$$(t, v(t)) \in B \ a.e. \text{ in } G, \qquad B \subset \mathbb{R}^{N+m}, \qquad v_{ij} = F_{ij}(t, u(t)).$$

Thus, $V(v_0) = V(u) \leq W_0$, and by the Lipschitzianity of F in (t, u) of costant C, we have $V(v^*) \leq CV(u) \leq CW_0$, and finally $V(v) \leq (C+1)W_0$ for $v \in \Omega'$. Since w is bounded on D, the mean values on G of the functions u are equibounded, or $|m.v.(\mathrm{u})| \leq \mathrm{K}$ for some constant K. Analogously, since w is bounded on D, and F is continuous, then $F(t, w(t))$ is bounded on D, and hence, the mean values on G of the functions v are equibounded, or $|\mathrm{m.n.}(v)| \leq K'$ for some constant K', because of the assumption at the beginning of Section 2. By Cafiero's and Fleming's compactness theorem the class Ω' is compact in L_1.

Thus, for any sequence $[v_k]$ in Ω', there is a subsequence, say still (k), such that $v_k \to v$ in L_1 with $v_k = \Phi u_k$, $u_k \in \Omega$. By the same theorem, there is a subsequence, say still $[k]$, such that $u_k \to u$ in L_1. Thus $v_k = (v_{k0}, v_k^*)$, $v_{k0} = u_k$, $v_k^*(t) = F(t, u_k(t))$, $t \in G$, and for

$$v^*(t) = F(t, u(t)), \qquad t \in G,$$

also

$$\|v_k^* - v^*\|_{L_1} = \|F(t, u_k(t) - F(t, u(t))\|_{L_1} \leq C\|u_k - u\|_{L_1},$$

that is, $v_k^* \to v^*$, $v_{k0} \to v_0$, or $v_k \to v$ in L_1. Thus, the class Ω' is relatively compact. The class Ω' is also closed in L_1. Indeed, if (v_k) is any sequence in Ω' with $v_k \to v$ in L_1, then, $v_k = (v_{k0}, v_k^*)$, $v = (v_0, v^*)$, $v_{k0} \to v_0$, $v_k^* \to v^*$ in L_1 with $v_{k0} = u_k$, $v_0 = u$, $u_k \to u$ in L_1.

Moreover,

$$\|v^*(t) - F(t, u(t))\|_{L_1} \leq \|v^* - v_k^*\|_{L_1} + \|v_k^* - F(t, u_k(t))\|_{L_1} + \|F(t, u_k(t)) - F(t, u(t))\|_{L_1},$$

where

$$\|v^* - v_k^*\|_{L_1} \to 0, \qquad \|v_k^* - F(t, u_k(t))\|_{L_1} = 0,$$
$$\|F(t, u_k(t)) - F(t, u(t))\|_{L_1} \leq C\|u_k - u\|.$$

Thus,

$$\|v^*(t) - F(t, u(t))\|_{L_1} = 0, \quad \text{or } v^*(t) = F(t, u(t)) \ a.e. \text{ in } G.$$

We have considered the functions $v = \Phi u$ for $u \in \Omega$, that is, the class Ω' of functions v, and we have seen that Ω' is closed in L_1.

By Theorem A we conclude that the Serrin-type integral $\mathbb{H}$ relative to H has an absolute minimum in Ω' given by some $v = (v_0, v^*)$, $v^*(t) = F(t, u(t))$, $v_0(t) = u(t)$, $t \in G$, $u \in \Omega$.

Hence, u gives the absolute minimum of the Serrin integral $\mathfrak{I}$ relative to I in Ω, and $0 \leq I(u) \leq \mathfrak{I}(u)$. □

REMARK. Whenever we can prove that $I(u) = 0$, that is, $0 = I(u) \leq \mathfrak{I}(u)$, for the optimal solution $u(t) = (u_1, \ldots, u_m)$, $t \in G$, $u \in \Omega$, of the integral $\mathfrak{I}$ associated to the integral $I(u)$, then from (2.1) we derive that $u(t)$ is a BV, possibly discontinuous, solution of the partial differential system

$$\sum_{j\in[J]_i} (F_{ij}(t,u))_{t_j} + F_i(t,u) = 0, \qquad t\in G\,(a.e.),\ \ i=1,\dots,m,$$
$$u(t) = w(t), \qquad t\in D\subset \partial G.$$

REMARK. The reduction of the integral $\mathfrak{J}$ to the integral $\mathcal{H}$ has allowed us to apply to $\mathcal{H}$ verbatim the existence theorem A we had proved in [5]. Elsewhere we shall follow the same line of argument in connection with other analogous lower semicontinuous Serrin type integrals and with more sophisticated assumptions.

3. A relevant situation

Borrowing from hyperbolic problems, let P denote a trapezoid of the tx-space $\mathbb{R}^{1+\nu}$, say for T, R, M given constants, $MT<R$,

$$P=[0\leqslant t\leqslant T, -R+Mt\leqslant x_j\leqslant R-Mt, j=1,\dots,\nu].$$
$$u(0,x)=w(x), \qquad x\in D=[(0,x)|-R\leqslant x_j\leqslant R, j=1,\dots,\nu].$$

In the definition of functions $u(t,x)=(u_1,\dots,u_m)$ of bounded variation (BV) on P let us always include D in the full measure set P-E on which we compute the total variations of u. Then, if w is bounded on D, say $|w(x)|\leqslant M'$, $x\in D$, and the function u with $u(0,x)=w(x)$ have equibounded total variations on P, say $V_0(u,P)\leqslant M''$, then

$$\iint_P u(t,x)\,dt\,dx \leqslant \iint_P |u(0,x)|\,dt\,dx + \iint_P |u(t,x)-u(0,x)|\,dt\,dx \leqslant$$
$$\leqslant T\int_{-R}^{R} |w(0,x)|\,dx + \int_{-R}^{R} V_t(x)\,dx \leqslant 2RTM' + M''.$$

In other words, boundedeness of w and equiboundedness of the total variations of the functions u (with the above convention) implies the equiboundedness of the values of the functions u.

Let $A=[(t,x,u)|(t,x)\in P, -K\leqslant u_i\leqslant K, i=1,\dots,m]$, let $F_{i0}=u_i$, and $F_{ij}(t,x,u)$, $j=1,\dots,\nu$, be given functions of class C^1 on A, and let $F_i(t,x,u)$ be given Lipschitzian functions on A. Now the integral I of §2 becomes

$$I(u)=\int_P \sum_{i=1}^{m}\left| u_{it} + \sum_{j=1}^{\nu}(F_{ij}(t,x,u))_{x_j} + F_i(t,x,u)\right| dt\,dx, \qquad dx=dx_1\dots dx_\nu,$$

Then the Serrin integral $\mathfrak{J}$ associated to I has an absolute minimum i given by a BV possibly discontinuous function $u(t,x)=(u_1,\dots,u_m)$, $(t,x)\in P$, and $0\leqslant I(u)\leqslant \mathfrak{J}(u)=i$. Whenever $I(u)=0$, then u is a solution of the Cauchy problem for the differential system

$$u_{it} + \sum_{j=1}^{\nu}(F_{ij}(t,x,u))_{x_j} + F_i(t,x,u)=0, \qquad i=1,\dots,m,\ \ (t,x)\in P(a.e.),$$
$$u_i(0,x)=w_i(x), \qquad -R\leqslant x_j\leqslant R,\ \ j=1,\dots,\nu,\ \ i=1,\dots,m.$$

Elsewhere we shall further study the integral $I(u)$.

EXAMPLE 1. As a simple example, we consider here the case where $m=1$, $\nu=1$, where t, x are the independent variables, and

$$I(u)=\int_P |u_t+(\tfrac{1}{2}u^2)_x|\,dt\,dx,$$

where P is the trapeze

$$P=[(t,x)|0\leq t\leq T, -R+Mt\leq x\leq R-Mt]\subset \mathbb{R}^2, \qquad MT<R,$$

with Cauchy data $u(0,x)=w(x)$, $-R\leq x\leq R$, $|w(x)|\leq M$. Here, there is only one function F_i, say $F_1=0$, and only one function F_{ij}, say $F_{11}=\frac{1}{2}u^2$.

Let $A=[(t,x,u)|(t,x)\in P, -K\leq u\leq K]\subset\mathbb{R}^3$.

We shall consider $I(u)$ in the class Ω of all scalar BV functions $u(t,x)$, $(t,x)\in P$, with $V_0(u)\leq W_0$ for some constant W_0 sufficiently large.

By Theorem B, the Serrin integral $\mathfrak{I}(u)$ associated to $I(u)$ has an absolute minimum u, with $0\leq I(u)\leq\mathfrak{I}(u)$. If $I(u)=0$, then u would be a solution of the hyperbolic equation with Cauchy data

$$u_t+uu_x=0, \qquad (t,x)\in P(a.e.), \qquad u(0,x)=w(x), \qquad -R\leq x\leq R.$$

Note that the function $v=(v_0,v_1)$ is now $v_0=u$, $v_1=\frac{1}{2}u^2$, and B is the set

$$B=[(t,x,v)|(t,x)\in P, v=(v_0,v_1), v_1=\tfrac{1}{2}v_0^2, v_0=u, -M\leq u\leq M]\subset\mathbb{R}^4$$

EXAMPLE 2. We consider here the integral

$$I(u)=\int_P\sum_{i=1}^m\left|u_{it}+\sum_{j=1}^\nu a_{ij}(t,x,u_i)u_{ix_j}-f_i(t,x,u)\right|dt\,dx,\quad dx=dx_1\dots dx_\nu, \tag{3.1}$$

where t is a scalar, $x=(x_1,\dots,x_\nu)$, $u=(u_1,\dots,u_m)$, where P is the trapezoid

$$P=[(t,x)|0\leq t\leq T, -R+Mt\leq x_j\leq R-Mt, j=1,\dots,\nu]\subset\mathbb{R}^{1+\nu},\quad MT<R,$$

with Cauchy data $u(0,x)=w(x)$, or $u_i(0,x)=w_i(x)$, $i=1,\dots,m$, for $-R\leq x_j\leq R$, $j=1,\dots,\nu$, and $|w_i(x)|\leq M$. We take

$$A=[(t,x,u)|(t,x)\in P, -K\leq u_i\leq K]\subset\mathbb{R}^{1+\nu+m}, i=1,\dots,m].$$

The integral $I(u)$ can be written in the form (2.1).

Indeed, if certain primitive

$$A_{ij}(t,x,u_i)=\int^{u_i} a_{ij}(t,x,\alpha)d\alpha, \qquad i=1,\dots,m,\ j=1,\dots,\nu,$$

are of class C^1, then

$$A_{ij,x_j}(t,x,u_i)=\int_0^{u_i} a_{ij,x_j}(t,x,\alpha)d\alpha,$$

$$(A_{ij}(t,x,u_i(t,x)))_{x_j}=a_{ij}(t,x,u_i(t,x))u_{ix_j}(t,x)+\int^{u_i(t,x)} a_{ij,x_j}(t,x,\alpha)d\alpha=$$

$$=a_{ij}(t,x,u_i(t,x))u_{ix_j}(t,x)+A_{ij,x_j}(t,x,u_i(t,x)),$$

and $I(u)$ becomes

$$I(u) = \int_P \sum_{i=1}^{m} \left| u_{it} + \sum_{j=1}^{\nu} (A_{ij}(t,x,u_i(t,x))_{x_j} + F_i(t,x,u) \right| dt\, dx,$$

$$F_i(t,x,u) = -\sum_{j=1}^{\nu} A_{ij,x_j}(t,x,u_i) - f_i(t,x,u).$$

Note that the functions $v = (v_0, v^*)$ are now

$$v_0 = u = (u_1, \dots, u_m), \qquad v_{ij} = A_{ij}(t,x,u_i), j = 1, \dots, \nu, \quad i = 1, \dots, m,$$

with $v(t,x) \in B \subset \mathbb{R}^{m+\nu}$,

$$B = [(t,x,v) | (t,x) \in P, \qquad v_0 = u = (u_1, \dots, u_m), \qquad v_{ij} = A_{ij}(t,x,u_i),$$
$$[j = 1, \dots, \nu, i = 1, \dots, m, u \in \mathbb{R}^m] \subset \mathbb{R}^{1+\nu+m+m}.$$

Under the assumptions of Theorem B, the Serrin-type integral $\mathfrak{J}(u)$ associated to $I(u)$ has an absolute minimum i given by a *BV* function $u \in \Omega$, and $0 \leq I(u) \leq \mathfrak{J}(u)$.

If $I(u) = 0$, then u is a solution of the Lax-type differential system with Cauchy data

$$u_{it} + \sum_{j=1}^{\nu} a_{ij}(t,x,u_i) u_{ix_j} = f_i(t,x,u), \qquad (t,x) \in P(a.e.), \qquad i = 1, \dots, m,$$

(3.2)
$$u(0,x) = w(x), \qquad -R \leq x_j \leq R, \qquad j = 1, \dots, \nu,$$
$$x = (x_1, \dots, x_\nu), \qquad u = (u_1, \dots, u_m).$$

For $m = 1$, (3.1) reduces to

$$I(u) = \int_P \left| u_t + \sum_{j=1}^{\nu} a_j(t,x,u) u_{x_j} - f(t,x,u) \right| dt\, dx,$$

where t and u are scalars, $x = (x_1, \dots, x_\nu)$, and P is as above. We take $A = [t,x,u) | (t,x) \in P, -M \leq u \leq M] \subset \mathbb{R}^{2+\nu}$. For the a_j all of class C^1,

$$A_j(t,x,u) = \int^u a_j(t,x,\alpha) d\alpha,$$

then $I(u)$ becomes

$$I(u) = \int_P \left| u_t + \sum_{j=1}^{\nu} (A_j(t,x,u))_{x_j} + F(t,x,u) \right| dt\, dx$$

where $F = -\sum_{j=1}^{\nu} A_{j,xj} - f$, and (3.2) reduces to

$$u_t + \sum_{j=1}^{\nu} a_j(t,x,u) u_{x_j} = f(t,x,u), \qquad (t,x) \in P.$$

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