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An existence and uniqueness result for Hamilton-Jacobi equations in Hilbert spaces

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Analisi matematica. — *An existence and uniqueness result for Hamilton-Jacobi equations in Hilbert spaces.* Nota di PIERMARCO CANNARSA e GIUSEPPE DA PRATO, presentata (*) dal Corrisp. R. CONTI.

ABSTRACT. — We prove an existence and uniqueness result for a class of Hamilton-Jacobi equations in Hilbert spaces.

KEY WORDS: Hamilton-Jacobi equations; Viscosity solutions; Optimal control.

RIASSUNTO. — *Un risultato di esistenza e unicità per equazioni di Hamilton-Jacobi in spazi di Hilbert.* Si dà un risultato di esistenza e unicità per un'equazione di Hamilton-Jacobi in spazi di Hilbert.

1. SETTING OF THE PROBLEM

Let X be a Hilbert space (norm $|\ |$, inner product $\langle , \rangle$). For any Hilbert space K and any nonnegative integer k we denote by $C^k(X,K)$ the set of all the mappings $f\colon X \to K$ which are continuous and bounded sets of X, together with their derivatives of order less than or equal to k.

We denote by $C^{k+1-}(X;K)$ the set of all mappings f in $C^k(X;K)$ whose derivative of order k is Lipschitz continuous on X.

We are here concerned with the following Hamilton-Jacobi equation:

$$(1)\qquad \begin{aligned} &-V_t(t,x) + H(B^* DV(t,x)) - \langle Ax + f(x), DV(t,x)\rangle = g(x) \\ &V(T,x) = \phi(x); \qquad\qquad x \in X;\quad t \in [0,T] \end{aligned}$$

under the following hypotheses (2):

i) $A\colon D(A) \subset X \to X$ generates an analytic semigroup e^{tA} in X. $\exists \omega \in \boldsymbol{R}$ such that $\|e^{tA}\| \leq e^{\omega t}$, $\forall t \geq 0$.

ii) The embedding $D(A) \to X$ is compact.

iii) $B \in \boldsymbol{L}(U;X)$ where U is another Hilbert space.

iv) $f \in C^{1-}(X;X)$.

v) g and ϕ belong to $C^{1-}(X:\boldsymbol{R})$ and are non-negative.

vi) $H(v) = \sup\{-\langle v,u\rangle - h(u); u \in U\}$, where $h \in C(U;\boldsymbol{R})$ is convex and such that $h(u) \geq c\|u\|^2$, $\forall u \in U$.

DEFINITION 1. We say that $V \in C([0,T] \times X; \boldsymbol{R})$ is a *generalized viscosity solution* of (1) if we have:

$$(3)\qquad \lim_{n\to\infty} V_n(t,x) = V(t,x), \qquad \forall x \in D(A),\ \forall t \in [0,T]$$

(*) Nella seduta del 22 giugno 1988.

where V_n is the viscosity solution of the problem:

$$(4)\qquad \begin{aligned} &-V_t^n(t,x) + H(B^* DV^n(t,x)) - \langle A_n x + f(x), DV^n(t,x)\rangle = g(x) \\ &V^n(T,x) = \Phi(x); \qquad\qquad x \in X;\ t \in [0,T] \end{aligned}$$

and $A_n = nA(n-A)^{-1}$ are the Yosida approximation of A.

We remark that, under hypotheses (2), problem (4) has a unique viscosity solution by the results of M. G. Crandall and P. L. Lions [4], [5] and [6]. We recall that $V^n \in C([0,T] \times X; \boldsymbol{R})$ is called a *viscosity solution* of (4) if, for any $(t,x) \in (0,T) \times X$ we have:

$$(5)\qquad \begin{aligned} &\text{i)}\ \forall (p_t, p_x) \in D^+ V(t,x),\quad -p_t + H(B^* p_x) - \langle A_n x + f(x), p_x\rangle \leqslant g(x) \\ &\text{ii)}\ \forall (p_t, p_x) \in D^- V(t,x),\quad -p_t + H(B^* p_x) - \langle A_n x + f(x), p_x\rangle \geqslant g(x) \end{aligned}$$

where the superdifferential D^+ and the subdifferential D^- are defined as:

$$(6)\qquad \begin{aligned} &D^+ V(t,x) = \\ &= \left\{(p_t, p_x) \in \boldsymbol{R} \times X;\ \lim \sup_{(s,y)\to(t,x)} \frac{V(s,y) - V(t,x) - (s-t)p_t - (y-x)p_x}{|s-t| + |y-x|} \leqslant 0\right\} \\ &D^- V(t,x) = \\ &= \left\{(p_t, p_x) \in \boldsymbol{R} \times X;\ \lim \inf_{(s,y)\to(t,x)} \frac{V(s,y) - V(t,x) - (s-t)p_t - (y-x)p_x}{|s-t| + |y-x|} \geqslant 0\right\}. \end{aligned}$$

In [4]-[6] it is also proved that $V^n(t,x)$ coincides with the value function:

$$(7)\qquad V^n(t,x) = \inf\left\{\int_t^T [g(y_n(s)) + h(u(s))]ds + \phi(y_n T); u \in L^2(t,T:U)\right\}$$

where:

$$(8)\qquad y_n'(s) = A_n y_n(s) + f(y_n(s)) + Bu(s), y_n(t) = x, \qquad t \leqslant s \leqslant T.$$

We remark that the definition of viscosity solution given by M. G. Crandall and P. L. Lions in [4]-[6] cannot be immediately extended to problem (1) due to the unboundedness of A. For this reason, M. G. Crandall and P. L. Lions in [7] have modified their definition to cover this case.

The main idea of the presented work consists of solving (1) in two steps. First we use the «classical» definition of viscosity solution of [4]-[6] for the approximating problem (4) in order to take care of the nonlinear term H. Then we follow a typical procedure of evolution equations theory to treat the difficulties coming from the unbounded operator A. This approach seems quite simple and does not require the self-adjointness of A.

2. The main result

Theorem. Under hypotheses (2) there exists a unique generalized viscosity solution V of problem (1).

Moreover,

(9) $$V(t,x) = \inf\left\{ \int_t^T [g(y_n(s)) + h(u(s))]ds + \phi y(T); u \in L^2(t, T: U)\right\}$$

where:

(10) $$y'(s) = Ay(s) + f(y(s)) + Bu(s), \quad y(t) = x; \qquad t \leqslant s \leqslant T.$$

PROOF. Uniqueness is a trivial consequence of Definition 1 and Theorem 1 in [4]. Let us prove existence. If V is given by (9), then, from V. Barbu and G. Da Prato [3] and V. Barbu [1] it follows that:

i) V is continuous in $[0, T] \times X$.

ii) $\forall (t,x) \in [0, T] \times X$, there exists an optimal pair (u^*, y^*).

Let $(t,x) \in [0, T] \times D(A)$ be fixed and (u^*, y^*) an optimal pair for V at (t,x). Let y_n be the solution of (8) with $u = u^*$. Clearly $y_n \to y^*$ in $C([t, T]; X)$. So

(11) $$V(t,x) = \lim_{n\to\infty} \int_t^T [g(y_n(s)) + h(u^*(s))]ds + \phi(y_n(T)) \geqslant \lim \sup_{n\to\infty} V^n(t,x).$$

Next we claim that

(12) $$\lim \inf_{n\to\infty} V^n(t,x) \geqslant V(t,x)$$

which will conclude our proof.

Assume (12) does not hold. Then there exists a subsequence, which is still denoted by $\{V^n(t,x)\}$, such that

(13) $$\lim_{n\to\infty} V_n(t,x) < V(t,x).$$

Let $u_n \in L^2 t, T; U)$ be such that

(14) $$V^n(t,x) + \frac{1}{n} > \int_t^T [g(z_n(s) + h(u_n(s))]ds + \phi(z_n(t))$$

where z_n is the solution of

(15) $$z_n'(s) = A_n z_n(s) + f(z_n(s)) + Bu_n(s), \quad z_n(t) = x; \qquad t \leqslant s \leqslant T.$$

From assumptions (2) and from (11) we conclude that $\{u_n\}$ is bounded in $L^2(t, T; U)$. Therefore, there exists a subsequence, still denoted by $\{u_n\}$, and a function $v \in L^2(t, T; U)$ such that

$$u_n \to v, \text{ weakly in } L^2(t, T; U).$$

Recalling (2) i), iv), from (15) it easily follows that $\{z_n\}$ in bounded in $L^\infty(t, T; U)$. Then, we can find a subsequence, still denoted by $\{z_2\}$, such that

$$z_n \to z, \qquad \zeta_n := f(z_z) + Bu_n \to \zeta, \qquad \text{weakly in } L^2(t, T: U).$$

Also, by (15)

$$z_n(s) = e^{(s-t)A_n}x + \int_t^s [e^{(s-\sigma)A_n} - e^{(s-\sigma)A}]\zeta_n(\sigma)d\sigma + \tag{16}$$

$$+ \int_t^s e^{(s-\sigma)A}\zeta_n(\sigma)d\sigma =: I_1^n(s) + I_2^n(s) + I_3^n(s).$$

Now,

$$I_1^n(s) \to e^{(s-t)A}x \quad \text{in } C([t,T];X) \tag{17}$$

and, since $\{\zeta_n\}$ is bounded in $L^2(t,T;X)$ and A generates an analytic semi-group,

$$I_2^n(s) \to 0, \qquad \forall s \in [t,T] \tag{18}$$

by the dominated convergence theorem. Finally,

$$I_3^n(s) \to \int_t^s e^{(s-\sigma)A}\zeta(\sigma)d\sigma, \qquad \forall s \in [t,T] \tag{19}$$

due to the compactness of the semigroup e^{tA}

Therefore, from (16)-(19)

$$\lim_{n\to\infty} z_n(s) = z(s) = e^{(s-t)A}x + \int_t^s e^{(s-\sigma)A}\zeta(\sigma)d\sigma, \qquad \forall s \in [t,T]. \tag{20}$$

Consequently

$$\lim_{n\to\infty} f(z_n(s)) = f(z(s)), \qquad \forall s \in [t,T]$$

and in fact z is the solution of

$$z'(s) = Az(s) + f(z(s)) + Bv(s), \quad z(t) = x; \qquad t \leqslant s \leqslant T.$$

Now, by the convexity of h and by (14) we obtain

$$\lim\inf_{n\to\infty} V^n(t,x) \geqslant \int_t^T [g(z(s)) + h(v(s))]ds + \varphi(z(T)) \geqslant V(t,x)$$

which contradict (13). The proof is thus complete.

REMARK. By variational arguments (see P. L. Lions [8] and V. Barbu [2]) one can show that the value function given by (9) satisfies (1) in the sense that for every $(t,x) \in (0,T) \times D(A)$ we have:

i) $\forall (p_t, p_x) \in D^+V(t,x), \qquad -p_t + H(B^*p_x) - \langle Ax + f(x), p_x \rangle \leqslant g(x)$

ii) $\forall (p_t, p_x) \in D^-V(t,x), \qquad -p_t + H(B^*p_x) - \langle Ax + f(x), p_x \rangle \geqslant g(x)$

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