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Injectors of fitting classes of $\mathfrak{G}_1$-groups

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Teoria dei gruppi. — *Injectors of fitting classes of* $\mathfrak{S}_1$*-groups* (*). Nota di FEDERICO MENEGAZZO e MARTIN L. NEWELL, presentata (**) dal Corrisp. G. ZACHER.

ABSTRACT. — Fitting classes and injectors are discussed in the class of $\mathfrak{S}_1$-groups. A necessary and sufficient condition for the existence of injectors is given; it is also shown that, when this condition holds, the injectors form a unique conjugacy class.

KEY WORDS: Soluble groups; Fitting classes; Injectors.

RIASSUNTO. — *Iniettori di classi di Fitting di gruppi* $\mathfrak{S}_1$. Si assegna una condizione necessaria e sufficiente per l'esistenza di iniettori per classi di Fitting nei gruppi $\mathfrak{S}_1$; se tale condizione è soddisfatta, gli iniettori risultano coniugati.

In recent years some attempts have been made to extend the beautiful theorem of [2] on existence and conjugacy of injectors from its original setting - finite soluble groups - to more general classes of soluble groups.

Interesting results on this line may be found in [4] and [1]. In particular [1] thoroughly investigates a slightly non-standard version of Fitting classes in $\mathfrak{S}_1$-groups, giving positive results (*i.e.* injectors exist and are conjugate) in many cases, and exhibiting some natural examples of groups and Fitting classes with no injectors.

The aim of our paper is to give necessary and sufficient conditions for the existence and conjugacy of injectors in $\mathfrak{S}_1$-groups. Namely, we will prove

THEOREM A. *Let $\mathfrak{X}$ be a Fitting class of $\mathfrak{S}_1$-groups. For a group $G \in \mathfrak{S}_1$ the following are equivalent:*

(i) *Every subgroup of G containing the $\mathfrak{X}$-radical $G_{\mathfrak{X}}$ has $\mathfrak{X}$-injectors;*
(ii) *G has a normal subgroup M containing $G_{\mathfrak{X}}$ such that:*

1) *every $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$ is contained in M, and*
2) *$M/G_{\mathfrak{X}}$ is (periodic abelian)-by-finite.*

THEOREM B. *Let $\mathfrak{X}$ be a Fitting class of $\mathfrak{S}_1$-groups. Assume $G \in \mathfrak{S}_1$ and all subgroups of G containing $G_{\mathfrak{X}}$ have $\mathfrak{X}$ injectors. Then in every such subgroup the $\mathfrak{X}$-injectors form a unique conjugacy class.*

In section 1 we fix the notation and collect some preparatory lemmas. Sections

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2 and 3 contain the proof of Theorems A and B. In section 4 we discuss the class $\mathfrak{N}$ of nilpotent $\mathfrak{S}_1$-groups.

1. Preliminaries

A Fitting class of $\mathfrak{S}_1$-groups is a subclass $\mathfrak{X}$ of $\mathfrak{S}_1$ satisfying

(1) if $G \in \mathfrak{X}$ and $S \lhd\lhd G$, then $S \in \mathfrak{X}$; and

(2) if $G \in \mathfrak{S}_1$ and G is generated by normal $\mathfrak{X}$-subgroups, then $G \in \mathfrak{X}$.

(See [3] for the definition of $\mathfrak{S}_1$).

If $\mathfrak{X}$ satisfies the stronger requirements

(A1) if $G \in \mathfrak{X}$ and A is an ascendant subgroup of G, then $A \in \mathfrak{X}$; and

(A2) if $G \in \mathfrak{S}_1$ and G is generated by ascendant $\mathfrak{X}$-subgroups, then $G \in \mathfrak{X}$

then we call $\mathfrak{X}$ an ascendant Fitting class (of $\mathfrak{S}_1$-groups). (Notice that our ascendant Fitting classes are called Fitting classes in [1]).

If $\mathfrak{X}$ is a Fitting class and $G \in \mathfrak{S}_1$, the $\mathfrak{X}$-radical $G_{\mathfrak{X}}$ of G is the subgroup generated by all normal $\mathfrak{X}$-subgroups of G. It is clear that $G_{\mathfrak{X}} \in \mathfrak{X}$; it is also clear that, if $\mathfrak{X}$ is an ascendant Fitting class, then $G_{\mathfrak{X}}$ is the subgroup generated by all ascendant $\mathfrak{X}$-subgroups of G.

Let $\mathfrak{X}$ be a Fitting class. We recall from [1] the definition of the characteristic of $\mathfrak{X}$:

char $\mathfrak{X} = \infty$ if $\mathfrak{X}$ contains an infinite cyclic group;

char $\mathfrak{X} = \{p : p$ a prime and $\mathfrak{X}$ contains a cyclic group of order $p\}$ otherwise.

The following lemma is a convenient reformulation of Section 2 in [1].

Lemma 1.1. *Let $\mathfrak{X}$ be a Fitting class.*

(a) *If $\mathfrak{X}$ contains a non-periodic group, then char $\mathfrak{X} = \infty$ and $\mathfrak{X}$ contains all nilpotent $\mathfrak{S}_1$-groups.*

(b) *If every group in $\mathfrak{X}$ is periodic, then every group in $\mathfrak{X}$ is a π-group where $\pi =$ char $\mathfrak{X}$, and $\mathfrak{X}$ contains all nilpotent $\mathfrak{S}_1$-π-groups.*

(c) *If $\mathfrak{X}$ is an ascendant Fitting class, then statements (a) and (b) hold with «locally nilpotent» replacing «nilpotent».*

Our definition of injectors is the standard one: if $G \in \mathfrak{S}_1$ and $\mathfrak{X}$ is a Fitting class, a subgroup V of G is an $\mathfrak{X}$-injector of G if $V \cap S$ is a maximal $\mathfrak{X}$-subgroup of S for every subnormal subgroup S of G. An $\mathfrak{X}$-injector V of G will be called ascendant-sensitive if $V \cap A$ is a maximal $\mathfrak{X}$-subgroup of A for every ascendant subgroup A of G.

The following lemmas will be useful in proving that a subgroup is an injector.

Lemma 1.2.

(a) *Let $\mathfrak{X}$ be Fitting glass. Assume W is an $\mathfrak{X}$-subgroup of S, and $S \lhd\lhd G$. If $W \geq S_{\mathfrak{X}}$ then $WG_{\mathfrak{X}} \in \mathfrak{X}$.*

(b) *Let $\mathfrak{X}$ be an ascendant Fitting class. Assume W is an $\mathfrak{X}$-subgroup of A, and A asc G. If $W \geqslant A_{\mathfrak{X}}$ then $WG_{\mathfrak{X}} \in \mathfrak{X}$.*

Proof. Part (b) is Lemma 4.2 of [1]. To prove part (a), induct on the subnormal defect of S. S is subnormal in $T := S^G$ whit smaller defect, so $V := WT_{\mathfrak{X}} \in \mathfrak{X}$ by induction. $[V, G_{\mathfrak{X}}] \leqslant T \cap G_{\mathfrak{X}} = T_{\mathfrak{X}} \leqslant V$, hence V and $G_{\mathfrak{X}}$ are normal $\mathfrak{X}$-subgroups of $VG_{\mathfrak{X}}$. It follows that $WG_{\mathfrak{X}} = VG_{\mathfrak{X}} \in \mathfrak{X}$.

Lemma 1.3. *Let V be a maximal $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$ where $\mathfrak{X}$ is a Fitting class.*

(a) *If $S \lhd\lhd G$ and $V \cap SG_{\mathfrak{X}}$ is a maximal $\mathfrak{X}$-subgroup of $SG_{\mathfrak{X}}$, then $V \cap S$ is a maximal $\mathfrak{X}$-subgroup of S.*

(b) *Assume further that $\mathfrak{X}$ is an ascendant Fitting class. If A asc G and $V \cap AG_{\mathfrak{X}}$ is a maximal $\mathfrak{X}$-subgroup of $AG_{\mathfrak{X}}$, then $V \cap A$ is a maximal $\mathfrak{X}$-subgroup of A.*

Proof of (a). Let W be an $\mathfrak{X}$-subgroup of S containing $V \cap S$. Since $W \geqslant V \cap S \geqslant G_{\mathfrak{X}} \cap S = S_{\mathfrak{X}}$, Lemma 1.2 gives $WG_{\mathfrak{X}} \in \mathfrak{X}$. $V \cap SG_{\mathfrak{X}} = G_{\mathfrak{X}}(V \cap S) \leqslant G_{\mathfrak{X}} W$ and maximality of $V \cap SG_{\mathfrak{X}}$ give $G_{\mathfrak{X}}(V \cap S) = G_{\mathfrak{X}} W$. Hence $W = W \cap G_{\mathfrak{X}}(V \cap S) = (V \cap S)(W \cap G_{\mathfrak{X}}) \leqslant (V \cap S)(S \cap G_{\mathfrak{X}}) = V \cap S$, so $V \cap S = W$ and $V \cap S$ is a maximal $\mathfrak{X}$-subgroup of S. (The proof of (b) is entirely similar; the extra assumption on $\mathfrak{X}$ is required to ensure $G_{\mathfrak{X}} \cap A = A_{\mathfrak{X}}$).

2. The necessary condition

We study first Fitting classes of periodic groups.

Proposition 2.1. *Let $\mathfrak{X}$ be a Fitting class of periodic $\mathfrak{S}_1$-groups. Let G be an $\mathfrak{S}_1$-group; further, let R denote the subgroup of G generated by all quasi-cyclic subgroups, F/R the Fitting subgroup of G/R, T/R the torsion subgroup of F/R. If all subgroups of G containing $G_{\mathfrak{X}}$ have $\mathfrak{X}$-injectors, then $[F, V] \leqslant T$ for every $\mathfrak{X}$-subgroup V of G containing $G_{\mathfrak{X}}$.*

Proof. R is an abelian divisible torsion group of finite rank (see [3], Thm. 10.33); hence $G_{\mathfrak{X}}$ contains the π-component of R, where $\pi = \operatorname{char}\mathfrak{X}$, by Lemma 1.1(b). If V is an $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$, then V is a Černikov π-group, again by Lemma 1.1 (b), and its finite residual is contained in $G_{\mathfrak{X}}$ by the remark above. It follows that $|V : G_{\mathfrak{X}}| < \infty$, and we may induct on $|V : G_{\mathfrak{X}}|$. Also $[G_{\mathfrak{X}}, F] \leqslant G_{\mathfrak{X}} \cap F \leqslant T$; so we may assume $V > G_{\mathfrak{X}}$. Let $W/G_{\mathfrak{X}}$ be a maximal normal subgroup of $V/G_{\mathfrak{X}}$; $W \in \mathfrak{X}$, $|V/W|$ is a prime, p say, and by induction we may assume $[F, W] \leqslant T$. Let $v \in V$ be such that $V/W = \langle vW \rangle$. Set $N := C_F(Soc(R))$; then N is locally nilpotent, $R \leqslant N$ and $|F : N|$ is finite (see [1], proof of Thm. 3.1.). Suppose $c \in N$ has $[c, v] \notin T$. $\langle c \rangle^V G_{\mathfrak{X}}/G_{\mathfrak{X}}$ is finitely generated, hence nilpotent; thus it has a torsion-free V-invariant normal subgroup $N_0/G_{\mathfrak{X}}$ of finite index. Now $[N_0, v] \nleqslant G_{\mathfrak{X}}$ (else $[c^k, v] \in G_{\mathfrak{X}} \cap N \leqslant T$ for some positive k, which is not the case), while $[N_0, W] \leqslant N_0 \cap [FG_{\mathfrak{X}}, W] \leqslant N_0 \cap TG_{\mathfrak{X}} = G_{\mathfrak{X}}$. Choose $a \in N_0$ such that $b = [a, v] \notin G_{\mathfrak{X}}$; set $B := \langle b \rangle^V G_{\mathfrak{X}} = \langle b \rangle^{\langle v \rangle} G_{\mathfrak{X}}$, and $Y := VB$; observe that $[B, v] \nleqslant G_{\mathfrak{X}}$ and $V \cap B = G_{\mathfrak{X}}$. Since $[B, W] \leqslant G_{\mathfrak{X}} \leqslant W$ we have $W \leqslant Y_{\mathfrak{X}}$;

and it actually is $Y_{\mathfrak{X}} = W$: else $Y = Y_{\mathfrak{X}} B$ ($Y_{\mathfrak{X}}$ is periodic, so that $Y_{\mathfrak{X}} \cap B = G_{\mathfrak{X}}$), and we would get $Y/Y_{\mathfrak{X}}$ torsion-free, $V \leqslant Y_{\mathfrak{X}}$ and the contradiction $[B, V] \leqslant G_{\mathfrak{X}}$. From the equality

$$(bv^{-1})^p = (a^{-1}v^{-1}a)^p = a^{-1}v^{-p}a \equiv v^{-p} \pmod{G_{\mathfrak{X}}}$$

we get

$$1 \equiv bb^v \dots b^{v^{p-1}} \equiv b^p \pmod{Y' G_{\mathfrak{X}}},$$

so that $Y/Y_{\mathfrak{X}} Y'$ is elementary, generated by the images of v and b. Since $(B/G_{\mathfrak{X}})^p < (B/G_{\mathfrak{X}})$ we have a noncyclic p-group as a factor group of $Y/Y_{\mathfrak{X}}$, so $Y/Y_{\mathfrak{X}} Y'$ is noncyclic of order p^2. By our assumptions Y has an $\mathfrak{X}$-injector X. $X \geqslant Y_{\mathfrak{X}}$, and the inclusion is proper because $Y_{\mathfrak{X}} = W$ is not a maximal $\mathfrak{X}$-subgroup of Y: thus $|X : Y_{\mathfrak{X}}| = p$. $X \cap VY'$ is then a maximal $\mathfrak{X}$-subgroup of $VY' \lhd Y$, hence again $X \cap VY'$ properly contains $Y_{\mathfrak{X}}$: but this just means $X \leqslant VY'$. The same argument yields $X \leqslant \langle bv^{-1} \rangle Y_{\mathfrak{X}} Y' = V^{a^{-1}} Y'$. Then $X \leqslant VY' \cap \langle bv^{-1} \rangle Y_{\mathfrak{X}} Y' = Y_{\mathfrak{X}} Y'$ and $X = Y_{\mathfrak{X}}$, a contradiction. So far, we proved $[N, V] \leqslant T$: since F/T is torsion-free nilpotent and NT/T has finite index in F/T, the desired conclusion $[F, V] \leqslant T$ follows.

Corollary 2.2. *Let $\mathfrak{X}$ be a Fitting class of periodic $\mathfrak{S}_1$-groups. Assume $G \in \mathfrak{S}_1$ and all subgroups of G containing $G_{\mathfrak{X}}$ have $\mathfrak{X}$-injectors. Then G has a normal Černikov subgroup M which contains all the $\mathfrak{X}$-subgroups of G containing $G_{\mathfrak{X}}$. In particular $M/G_{\mathfrak{X}}$ is (periodic abelian)-by-finite*

Proof. In the notation of 2.1, set $C = C_G(F/T)$; then C contains every $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$. $D = C_C(T/R)$ has finite index in C (this is because T/R is finite) and D stabilizes the chain $F/R \geq T/R \geq R/R$. F/R is the Fitting subgroup of G/R, so $D \leq F$ follows; in particular $|C : F \cap C|$ is finite. In the group C/T the subgroup $F \cap C/T$ is contained in the centre; hence C/T is centre-by-finite and its periodic elements form a finite subgroup M/T. As every $\mathfrak{X}$-group is periodic, it follows that M has all the required properties.

If we strengthen somewhat our hypotheses, assuming that every subgroup of a given $\mathfrak{S}_1$-group has $\mathfrak{X}$-injectors, then we get a nicer result.

Proposition 2.3. *Let $\mathfrak{X}$ be a Fitting class of periodic $\mathfrak{S}_1$-groups. Assume $G \in \mathfrak{S}_1$ and all subgroups of G have $\mathfrak{X}$-injectors. Then G has a normal Černikov subgroup M which contains all the π-elements of G, where $\pi = \operatorname{char} \mathfrak{X}$.*

Proof. We stick to the notation of 2.1. Let x be a π-element of G; we will show that $[F, x] \leqslant T$. We may assume that $|x| = p^n$ for some prime $p \in \pi$, and that $[F, x^p] \leqslant T$. Suppose there is $c \in N$ such that $[c, x] \notin T$. The finitely generated nilpotent group $\langle c \rangle^{\langle x \rangle}$ has a torsion-free normal x-invariant subgroup N_0 of finite index, and $[N_0, x] \neq 1$. Again choose $a \in N_0$ with $b = [a, x] \neq 1$; set $B := \langle b \rangle^{\langle x \rangle}$ and $Y := \langle x \rangle B$. Every finite p-group is in $\mathfrak{X}$; so the same argument we had in the proof of 2.1 now shows $Y_{\mathfrak{X}} = \langle x^p \rangle$, $Y/Y_{\mathfrak{X}} Y'$ elementary of order p^2, generated by the images of x and b and, if X is any $\mathfrak{X}$-injector of Y, then $|X/Y_{\mathfrak{X}}| = p$. But then $X \cap \langle x \rangle Y'$, which is a maximal $\mathfrak{X}$-subgroup of $\langle x \rangle Y'$, properly contains $Y_{\mathfrak{X}}$, so that $X \leqslant \langle x \rangle Y'$, and, by the same argument, $X \leqslant \langle bx^{-1} \rangle Y' = \langle x \rangle^{a^{-1}} Y'$. This forces $X = Y_{\mathfrak{X}}$, a contradiction. Now we have $[N, x] \leqslant T$, and again $[F, x] \leqslant T$ follows easily. But this also proves that the subgroup M, as defined in the proof of 2.2, will contain every π-element of G.

Next we look at non-periodic Fitting classes; the analogue of Proposition 2.1 is the following

PROPOSITION 2.4. *Let $\mathfrak{X}$ be a Fitting class of $\mathfrak{S}_1$-groups. Assume $G \in \mathfrak{S}_1$ and all subgroups of G containing the radical $G_{\mathfrak{X}}$ have $\mathfrak{X}$-injectors. Suppose in addition that G has a normal subgroup A such that $G_{\mathfrak{X}} \leqslant A$, $A/G_{\mathfrak{X}}$ is abelian and $|G/A|$ is finite; let $T/G_{\mathfrak{X}}$ be the torsion subgroup of $A/G_{\mathfrak{X}}$. If V is any $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$, then $[V, A] \leqslant T$.*

PROOF. For every such subgroup V we have $V \cap A = G_{\mathfrak{X}}$ and $|V/G_{\mathfrak{X}}|$ is finite; so we may use induction on $|V/G_{\mathfrak{X}}|$. Let $W/G_{\mathfrak{X}}$ be a maximal normal subgroup of $V/G_{\mathfrak{X}}: V/W$ is cyclic of some prime order p, say $V/W = \langle vW \rangle$. $W \in \mathfrak{X}$; by induction $[W, A] \leqslant T$, and we assume $[V, A] \nleqslant T$, so that there exists $c \in A$ with $[c, v] \notin T$. The finitely generated abelian subgroup $\langle c \rangle^V G_{\mathfrak{X}}/G_{\mathfrak{X}}$ of $A/G_{\mathfrak{X}}$ has a nontrivial V-invariant free abelian subgroup $A_0/G_{\mathfrak{X}}$ of finite index, and $[A_0, W] \leqslant G_{\mathfrak{X}} = T \cap A_0$ while $[A_0, v] \nleqslant G_{\mathfrak{X}}$. Arguing as in the proof of Proposition 2.1, choose $a \in A_0$ such that $b = [a, v] \notin G_{\mathfrak{X}}$, set $B := \langle b \rangle^V G_{\mathfrak{X}}/G_{\mathfrak{X}}$ and $Y := VB$. Then $B/G_{\mathfrak{X}}$ is free abelian and $[B, v] \nleqslant G_{\mathfrak{X}}$. Now $[B, W] \leqslant G_{\mathfrak{X}} \leqslant W$ gives $W \lhd Y$, so that $W \leqslant Y_{\mathfrak{X}}$. In fact $W = Y_{\mathfrak{X}}$, else we would have $Y = Y_{\mathfrak{X}} B$, $Y/G_{\mathfrak{X}} = Y_{\mathfrak{X}}/G_{\mathfrak{X}} \times B/G_{\mathfrak{X}}$ and the contradiction $[V, B] \leqslant G_{\mathfrak{X}}$. From the equality

$$(bv^{-1})^p G_{\mathfrak{X}} = v^{-p} G_{\mathfrak{X}}$$

we get

$$b^p \equiv b^{1+v+\dots v^{p-1}} \equiv 1 \ (\mathrm{mod}\, Y' G_{\mathfrak{X}})\,;$$

the group Y/Y' is then elementary abelian, of order p^2, generated by the images of v and b (it cannot be cyclic, because $(B/G_{\mathfrak{X}})^p < B/G_{\mathfrak{X}}$). Now let X be an $\mathfrak{X}$-injector of Y; $X \geqslant Y_{\mathfrak{X}} = W$ and the inclusion is proper, since W is not a maximal $\mathfrak{X}$-subgroup of Y: it follows $|X/W| = |XB/WB| = p$. Once again, intersecting X with the normal subgroups VY' and $V^{a^{-1}} Y' = \langle bv^{-1} \rangle WY'$ of Y, we can conclude that $X \leqslant VY' \cap V^{a^{-1}} Y' = WY'$; but then we reach the contradiction $X = W$.

COROLLARY 2.5. *Let $\mathfrak{X}$ be a Fitting class of $\mathfrak{S}_1$-groups, with char $\mathfrak{X} = \infty$. Assume $G \in \mathfrak{S}_1$ and all subgroups of G containing $G_{\mathfrak{X}}$ have $\mathfrak{X}$-injectors. Then G has a normal subgroup M containing $G_{\mathfrak{X}}$ such that*

1) *every $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$ is contained in M; and*
2) *$M/G_{\mathfrak{X}}$ is (periodic abelian)-by finite.*

PROOF. By a theorem of Malcev's (see [3], Thm. 3.25) G is nilpotent-by-abelian-by-finite; Lemma 1.1 (a) then shows that G has a normal subgroup A with $G_{\mathfrak{X}} \leqslant A$, $A/G_{\mathfrak{X}}$ abelian and G/A finite. Set $C := C_G(A/T)$, where $T/G_{\mathfrak{X}}$ is the torsion subgroup of $A/G_{\mathfrak{X}}$. $C \geqslant A$, C/A finite and A/T central in C/T imply that C/T has finite torsion subgroup M/T. It is clear that M satisfies 2). If V is an $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$ then $V \leq C$ by Proposition 2.4. Moreover $V \cap A = G_{\mathfrak{X}}$ so that $V/G_{\mathfrak{X}}$, hence VT/T is finite. It follows $VT/T \leq M/T$, or $V \leq M$, proving 1).

3. The sufficient condition

It will be convenient to deal first with a special case.

Proposition 3.1. *Let $\mathfrak{X}$ be a Fitting class of $\mathfrak{S}_1$-groups. Assume $G \in \mathfrak{S}_1$ and $G/G_{\mathfrak{X}}$ is (periodic abelian)-by-finite. Then G has a unique conjugacy class of $\mathfrak{X}$-injectors.*

Proof. Let A be a normal subgroup of G such that $G_{\mathfrak{X}} \leq A$, $A/G_{\mathfrak{X}}$ is periodic abelian, and G/A is finite. Notice that if Y is any $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$ we have $Y \cap A = G_{\mathfrak{X}}$, and so $|Y/G_{\mathfrak{X}}|$ is finite, at most $|G/A|$ (as a consequence, the existence of maximal $\mathfrak{X}$-subgroups is not a problem). If $A = G$, then $G_{\mathfrak{X}}$ is the unique $\mathfrak{X}$-injector of G; and we may proceed by induction on $|G/A|$. Let M be a maximal normal subgroup of G containing A; by induction M has a unique class of $\mathfrak{X}$-injectors. Let U be an $\mathfrak{X}$-injector of M; U contains $M_{\mathfrak{X}} = G_{\mathfrak{X}}$ and U is contained in a maximal $\mathfrak{X}$-subgroup V of G. We claim that V is an $\mathfrak{X}$-injector of G. Let S be a subnormal subgroup of G: we have to show that $V \cap S$ is a maximal $\mathfrak{X}$-subgroup of S. By Lemma 1.3 we may assume $G_{\mathfrak{X}} \leq S$. Moreover, by a standard induction argument we can reduce to the case: $S \lhd G$, G/S abelian. Now we set $L := U \cap S$ and $N := N_G(L)$. Since $S \cap M \lhd M$ (in fact, $S \cap M \lhd G$ and $G/S \cap M$ is abelian) and $L = U \cap (S \cap M)$, L is an $\mathfrak{X}$-injector of $S \cap M$. From $U = M \cap V \trianglelefteq V$ we get $V \leq N$. Let W be a maximal $\mathfrak{X}$-subgroup of S containing $V \cap S$; then $L \leq W \cap M \leq S \cap M$. $W \cap M \in \mathfrak{X}$, so we have $W \cap M = L \trianglelefteq W$, or $W \leq N$; let Z be a maximal $\mathfrak{X}$-subgroup of N containing W, and set $H := \langle V, Z \rangle$. $H \leq N$, $N/G_{\mathfrak{X}}$ is locally finite, $V/G_{\mathfrak{X}}$ and $Z/G_{\mathfrak{X}}$ are both finite, hence $H/G_{\mathfrak{X}}$ is finite. Thus, we have $L \lhd H$, $L \in \mathfrak{X}$, H/L finite. Moreover, $L \leq S \cap M \cap H \trianglelefteq H$ and $H/S \cap M \cap H \cong H(S \cap M)/S \cap M \leq G/S \cap M$ is abelian. V and Z are maximal $\mathfrak{X}$-subgroups of H, both containing L; so Hartley's Lemma as extended in [1, Lemma 4.1] applies, and $Z = V^h$ for some $h \in H$. Notice that $Z \cap S$ is an $\mathfrak{X}$-subgroup of S containing W, so that $Z \cap S = W$. But then $W = Z \cap S = V^h \cap S = (V \cap S)^h$, and this implies $|W/G_{\mathfrak{X}}| = |V \cap S/G_{\mathfrak{X}}|$ and finally $V \cap S = W$, a maximal $\mathfrak{X}$-subgroup of S. This proves our first claim, *i.e.* G has $\mathfrak{X}$-injectors.

Let now V_1, V_2 be $\mathfrak{X}$-injectors of G; $V_1 \cap M$, $V_2 \cap M$ are both $\mathfrak{X}$-injectors of M, so conjugate in M. We may (and will) assume $V_1 \cap M = V_2 \cap M = U$, with U an $\mathfrak{X}$-injector of M. Now $G_{\mathfrak{X}} \leq U \leq V_i$ and $V_i/G_{\mathfrak{X}}$ is finite ($i = 1,2$), hence $K/G_{\mathfrak{X}}$ is finite, where $K := \langle V_1, V_2 \rangle$. $M \cap K \trianglelefteq K$, $K/M \cap K$ is abelian; U is a maximal $\mathfrak{X}$-subgroup of M and $U \leq K$, so that U is a maximal $\mathfrak{X}$-subgroup of $M \cap K$; V_1, V_2 are maximal $\mathfrak{X}$-subgroups of K containing U. Hartley's Lemma (as extended in [1]) applies again, giving V_1, V_2 conjugate in K.

Corollary 3.2. *Let $\mathfrak{X}$ be a Fitting class of $\mathfrak{S}_1$-groups. Assume $G \in \mathfrak{S}_1$ and G has a normal subgroup M containing $G_{\mathfrak{X}}$ such that*

1) *every $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$ is contained in M; and*
2) *$M/G_{\mathfrak{X}}$ is (periodic abelian)-by-finite.*

Then G has a unique conjucacy class of $\mathfrak{X}$-injectors: namely, the $\mathfrak{X}$-injectors of M.

PROOF. It is clear that, under the above assumptions, any $\mathfrak{X}$-injector of G will be an $\mathfrak{X}$-injector of M. To prove the converse, let V be an $\mathfrak{X}$-injector of M (they exist and are conjugate in M by Proposition 3.1); then $V \geq M_{\mathfrak{X}} = G_{\mathfrak{X}}$. Let $S \lhd\lhd G$ with $G_{\mathfrak{X}} \leq S$ (this is no restriction, in view of Lemma 1.3). If X is an $\mathfrak{X}$-subgroup of S containing $G_{\mathfrak{X}}$ we have $X \leq M \cap S$; thus the maximal $\mathfrak{X}$-subgroups of S containing $G_{\mathfrak{X}}$ are precisely the maximal $\mathfrak{X}$-subgroups of $M \cap S$ containing $G_{\mathfrak{X}}$. Now $V \cap (M \cap S)$ is a maximal $\mathfrak{X}$-subgroup of $M \cap S$ containing $G_{\mathfrak{X}}$ so $V \cap S = V \cap M \cap S$ is a maximal $\mathfrak{X}$-subgroup of S. This proves that V is an $\mathfrak{X}$-injector of G.

It is now a straightforward matter to prove Theorems A and B. Corollaries 2.2 and 2.5 show that statement (i) of Theorem A implies (ii) of the same theorem. Assume now that G and X satisfy (ii), and let H be a subgroup of G containing $G_{\mathfrak{X}}$. Then $H_{\mathfrak{X}} \geq G_{\mathfrak{X}}$, and every $\mathfrak{X}$-subgroup of H containing $H_{\mathfrak{X}}$ is contained in $M \cap H$; moreover, $M \cap H/H_{\mathfrak{X}}$ is (periodic abelian)-by-finite, so Corollary 3.2 applies, and H has $\mathfrak{X}$-injectors. This concludes the proof of Theorem A. To prove Theorem B, to each subgroup H of G containing $G_{\mathfrak{X}}$ first apply Theorem A (in the direction (i)$\rightarrow$(ii)), and then again Corollary 3.2.

In this context, a natural question is whether, in case the Fitting class $\mathfrak{X}$ is ascendant, any $\mathfrak{X}$-injector V of a group $G \in \mathfrak{S}_1$ will be ascendant-sensitive. The answer to this question is, in general, negative, as the following example shows. If p is any prime, let A be the free module with basis $e_1, \ldots, e_{p-1}$ over the ring $Z^{(p)}$ of rational numbers with denominator a power of p; denote by B the (free abelian) subgroup of A, generated by $e_1, \ldots, e_{p-1}$. Define the automorphism α of A by setting $e_i\alpha = e_{i+1}$ $(i = 1, \ldots, p-2)$, $e_{p-1}\alpha = -(e_1 + \ldots + e_{p-1})$; it is clear that $1 + \alpha + \ldots + \alpha^{p-1} = 0$, and that $B = B\alpha$. Finally, let $G := A\langle t \rangle$ be the split extension of A by a cyclic group of order p, with t acting on A as α does. The only proper subnormal subgroups of G are the subgroups of A, so that $\langle t \rangle$ is an $\mathfrak{X}$-injector of G if $\mathfrak{X}$ is the class of Černikov p-groups (or, for that matter, any ascendant Fitting class of periodic groups with $p \in \operatorname{char} \mathfrak{X}$). $H := B\langle t \rangle$ is an ascendant subgroup of G, and $H/H' = \langle e_1 H', tH' \rangle$ is elementary abelian of order p^2. It is now clear that $K := \langle e_1 t, H' \rangle$ asc G, but $K \cap \langle t \rangle = 1$ is not a maximal $\mathfrak{X}$-subgroup of K, since $\langle e_1 t \rangle \in \mathfrak{X}$. Notice also that H contains the $\mathfrak{X}$-injector $\langle t \rangle$ of G, but H has no $\mathfrak{X}$-injectors.

However, if $\mathfrak{X}$ is an ascendant Fitting class and char $\mathfrak{X} = \infty$, then any $\mathfrak{X}$-injector of a group $G \in \mathfrak{S}_1$ is ascendant-sensitive. In fact, we may restrict to ascendant subgroups containing $G_{\mathfrak{X}}$ by Lemma 1.3 (b); and these are subnormal in G because $G/G_{\mathfrak{X}}$ is polycyclic ([1] Thm. 3.1). The same conclusion holds if $\mathfrak{X}$ is an ascendant Fitting class of periodic $\mathfrak{S}_1$-groups, $G \in \mathfrak{S}_1$ and all subgroups of G containing $G_{\mathfrak{X}}$ have $\mathfrak{X}$-injectors: to prove this statement, apply theorem A and then Theorem 4.8 of [1].

4. Nilpotent $\mathfrak{S}_1$-groups

In this section we study some properties of the class $\mathfrak{N}$ of nilpotent $\mathfrak{S}_1$-groups (the symbol $\mathfrak{N}$ has been used in [1] to denote the class of locally nilpotent $\mathfrak{S}_1$-groups; we hope that no confusion will arise from this).

Lemma 4.1. *Let A be an abelian normal subgroup of the nilpotent group H. Assume A has finite rank r, and A is either* (i) *an elementary p-group; or* (ii) *a divisible p-group; or* (iii) *torsion-free. Then $[A, rH] = 1$.*

Proof. Case (i) is clear. For case (ii), if $A \neq 1$, then $[A, H]$ is a proper divisible subgroup of A, whose rank is strictly less than r; induction on the rank finishes this case. In case (iii), put $Z_i := Z_i(H) \cap A$. Z_{i+1}/Z_i is torsion-free: this is clear if $i = 0$; suppose Z_i/Z_{i-1} is torsion-free: if $a \in Z_{i+1}$ and $a^n \in Z_i$ then $[a^n, h] \equiv 1 \equiv [a, h]^n \pmod{Z_{i-1}}$ with $[a, h] \in Z_i$ for all $h \in H$, and this implies $[a, h] \in Z_{i-1}$, $a \in Z_i$. In particular A/Z_1 is torsion-free, its rank is strictly smaller than r (if $A \neq 1$): induction gives the conclusion in this case too.

Lemma 4.2. *Let G be an $\mathfrak{S}_1$ group. There is a natural number c such that for all nilpotent normal subgroups N of G $\gamma_{c+1}(N) = 1$.*

Proof. G has a finite series $G = G_0 > G_1 > \ldots > G_n = 1$ with $G_i \trianglelefteq G$ whose factors G_i/G_{i+1} are abelian of finite rank r_i, and either elementary p-groups, or divisible p-groups, or torsion-free. Fix i, and set $A := G_i/G_{i+1}$, $M := NG_{i+1}/G_{i+1}$. A and M are nilpotent normal subgroups of G/G_{i+1}; thus $H := AM$ is nilpotent and $[A, r_i M] = 1$ by Lemma 4.1. It follows that $[G_i, r_i N] \leqslant G_{i+1}$; if $c = \Sigma r_i$, then $\gamma_{c+1}(N) \leqslant [G, cN] = 1$.

The following corollaries, whose proof is straightforward after the lemmas above, might be well-known.

Corollary 4.3. *If $G \in \mathfrak{S}_1$ then Fit $G := \langle N : N \trianglelefteq G,\ N$ nilpotent $\rangle$ is nilpotent.*

Corollary 4.4. *The class $\mathfrak{N}$ of nilpotent $\mathfrak{S}_1$-groups is a Fitting class of $\mathfrak{S}_1$-groups.*

The proof or our next proposition is modeled after [4].

Proposition 4.5. *Let G be an $\mathfrak{S}_1$-group; set $F :=$ Fit G. There is a normal subgroup M of G such that $F \leqslant M$, M/F is finite, and all nilpotent subgroups of G containing F lie in M. Hence G has a unique class of $\mathfrak{N}$-injectors.*

Proof. Let R be the subgroup generated by all quasicyclic subgroups of G. R is divisible abelian of finite rank, $R \leq T \leq F$ where T is the torsion subgroup of F, and T/R is finite (see [3], Thm.10.33). R has a finite chain $R = R_0 \geqslant R_1 \geqslant \ldots \geqslant R_n = 1$ with $R_i \leqslant G$, and R_i/R_{i+1} abelian, divisible p-group, central in F (all i). The upper central series of F/T gives a finite chain $T = Z_0 \leqslant Z_1 \leqslant \ldots \leqslant Z_m = F$ with $Z_i \trianglelefteq G$ and Z_i/Z_{i-1} torsion-free abelian, central in F. Define M to be the intersection of the stabilizers of the chains $F = Z_m \geq \ldots \geq Z_1 \geq Z_0 = T$, $R = R_0 \geq R_1 \geq \ldots \geq R_n = 1$. Then $F \leq M$ and $F \geq C_M(T/R)$, so that M/F is finite. If X is any nilpotent subgroup of G containing F, then X/F is finite (since G/F is abelian-by-finite); it follows that X centralizes every R_i/R_{i+1} and also every Z_i/Z_{i-1}, so that $X \leq M$. The last statement of our proposition is now an immediate consequence of Theorems A and B.

The class $\mathfrak{N}$ is of course not ascendant. Moreover, $\mathfrak{N}$-injectors need not be ascendant-sensitive: if G is any non-nilpotent Černikov p-group, then its Fitting subgroup F is the unique $\mathfrak{N}$-injector of G; if H is a finite subgroup of G not contained in F, we have $H \operatorname{asc} G$ and $F \cap H$ is not a maximal $\mathfrak{N}$-subgroup of H.

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