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Algebra. — *The importance of rational extensions.* Nota di FRANS LOONSTRA, presentata (*) dal Socio G. ZAPPA.

ABSTRACT. — The rational completion $\overline{M}$ of an R-module M can be characterized as a τ_M-injective hull of M with respect to a (hereditary) torsion functor τ_M depending on M. Properties of a torsion functor depending on an R-module M are studied.

KEY WORDS: Torsion-functor; Rational extension.

RIASSUNTO. — *Funtore-torsione τ dipendente di un R-modulo M.* Si considerano le estensioni razionali e i completamenti razionali degli R-moduli. Il completamento razionale $\overline{M}$ di un R-modulo M può essere considerato come l'inviluppo τ-iniettivo $\overline{M} = M_\tau$ di M per uno speciale funtore-torsione τ dipendente da M. Vengono investigate le proprietà di τ.

1. INTRODUCTION

In the following a rational extension M of a non-zero submodule N (of M) will be the leading notion; M is a *rational extension* of N ($N \subseteq_r M$) if for any $m_1 \in M$, $0 \neq m_2 \in M$, $\exists r \in R$, such that $rm_1 \in N$, $rm_2 \neq 0$. We have the following equivalent statements: (i) $N \subseteq_r M$; (ii) $\mathrm{Hom}_R(A/N; M) = 0$ whenever $N \subseteq A \subseteq M$; (iii) $\mathrm{Hom}_R(M/N; \hat{M}) = 0$. The R-module M is called *rationally complete* if M has no proper rational extension. E.g. an injective R-module is rationally complete. Every R-module M has a rational extension $\overline{M} = E_r(M)$ which is rationally complete; $\overline{M}$ is unique up to

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isomorphism over M. We have the following representations of $\overline{M}$ (see [1]):

$$\overline{M} = \cap \{\ker\phi | \phi \in \mathrm{End}_R(\hat{M}); \phi(M) = 0\} = \{x \in \hat{M} | \forall 0 \neq y \in \hat{M}, \exists r \in R, rx \in M, ry \neq 0\}.$$

The rational completion $\overline{M}$ of an R-module M can be characterized as the τ injective hull M_τ (of M) for a special torsion functor (depending on M).

Any torsion functor τ to the category R-Mod can be defined by means of a *filter* $\boldsymbol{L}$ of left ideals of R (see e.g. [2]). If $\boldsymbol{L}$ is such a filter, then $\boldsymbol{L}$ defines for every R-module A a torsion submodule

$$\tau(A) = \tau_L(A) = \{a \in A | \mathrm{Ann}_R(a) \in \boldsymbol{L}\}; \tag{1}$$

conversely any torsion functor τ determines uniquely the corresponding filter $\boldsymbol{L}_\tau$ by $\boldsymbol{L}_\tau = \{I \subseteq_R R | R/I \text{ is } \tau\text{-torsion}\}$. A is called a τ_L *torsion* R-module if $\tau_L(A) = A$, and A is τ_L-*torsionfree* if $\tau_L(A) = 0$.

An R-module A is called τ-*injective*, if for every diagram

$$(*) \qquad \begin{array}{ccc} 0 \to & C & \to B \\ & f \downarrow & \swarrow f' \\ & A & \end{array}$$

with B/C being τ-torsion, any R-homomorphism $f: C \to A$ has an extension $f': B \to A$ making the diagram (*) commutative.

1.1. If A is any R-module, then A has a minimal τ-injective extension A_τ, $A \subseteq A_\tau \subseteq \hat{A}$, uniquely determind by the properties: (i) $\hat{A}/A_\tau$ is τ-torsionfree; (ii) $A_\tau/A \cong \tau(\hat{A}/A)$; (iii) $A \subseteq_e A_\tau$; (iv) $A_\tau = \{x \in \hat{A} | (A : x) \in \boldsymbol{L}_\tau\}$. The minimal τ-injective extension A_τ of A is called the τ-*injective hull* of A.

For more details about τ-injectivity, see [2]. In connection with the theory of rationals we need a special torsion functor τ_M defined by means of the *fixed* chosen R-module M and the corresponding filter

$$\boldsymbol{L}_M = \{I \subseteq_R R | \mathrm{Hom}_R(R/I; \hat{M}) = 0\}. \tag{2}$$

Then the torsion functor τ_M belonging to (2) is given in (1):

$$\tau_M(A) = \{a \in A | \mathrm{Hom}_R(R/\mathrm{Ann}(a); \hat{M}) = 0\,. \tag{3}$$

This implies, that an R-module A is τ_M-torsion if and only if

$$\mathrm{Hom}_R(A; \hat{M}) = 0.$$

1.2. Let $\overline{M}$ be the rational completion of the R-module M; then (i) $\overline{M}$ is the τ_M-injective hull of M; (ii) $\overline{M} = \{x \in \hat{M} | (M : x) \in \boldsymbol{L}_M\}$.

Pr.: For the proof we use the equivalent statements: (1) $\overline{M}$ is a rational extension of M; (2) $M \subseteq_e \overline{M}$ and $\overline{M}/M$ is τ_M-torsion. Now $\overline{M}$ satisfies the properties of the τ_M-injctive hull of M, and we have $\overline{M} \subseteq \hat{M}$. Let $K/\overline{M}$ be the τ_M-torsion submodule of $\hat{M}/\overline{M}$ then K is a rational extension of $\overline{M}$; since $\overline{M}$ is rationally complete, $K = \overline{M}$, and $\hat{M}/\overline{M}$ is τ_M-torsionfree, and therefore $\overline{M}$ is the τ_M-injective hull of M (in $\hat{M}$). Then 1.1 (iv) learns that the rational completion $\overline{M}$ of M is just the τ_M-injective hull of the R-module M.

1.3 Corollary. If M is a rationally complete R-module, then M is a τ_M-injective R-module.

2. THE HEREDITARY TORSION FUNCTOR τ_M

We want to study the family of all torsion theories τ of R-Mod for which $\tau(M)=0$ for a *fixed chosen* R-Module M. If $\tau(M)=0$ and $M\subseteq_e M'$, then $\tau(M')=0$.

Let $\Lambda(M)$ be the family of (hereditary) torsion functors τ with $\tau(M)=0$; then for $M\subseteq_e M'$ we have $\Lambda(M)=\Lambda(M')$; consequently, for a given module M, the set $\Lambda(M)$ can be obtained by using for M the injective hull $\hat{M}$ of M.

To avoid triviality we note that $\Lambda(0)$ is the collection of all torsion functors.

2.1. Let $M\neq 0$ be a fixed chosen R-module, A any R-module,

$$\tau_M(A)=\bigcap\{\ker(\phi|\phi\in \mathrm{Hom}_R(A;\hat{M})\},\tag{1}$$

then (i) τ_M is a hereditary torsion functor, $\tau_M(M)=0$, (ii) $\rho\in\Lambda(M)$, if and only if $\rho\leqslant\tau_M$.

If we define that the functor τ_2 is «stronger» than $\tau_1(\tau_1<\tau_2)$, if $\tau_1(A)\subset\tau_2(A)(\forall A\in R\text{-Mod})$, then this is equivalent with the property $\boldsymbol{L}_{\tau_1}\subset\boldsymbol{L}_{\tau_2}$ for the corresponding idempotent filters; we also say then: τ_1 is «weaker» than τ_2. Therefore the property (ii) expresses that, for a fixed chosen R-module M, τ_M is the strongest torsion functor ρ of R-Mod with the property $\rho(M)=0$. In other words: $\Lambda(M)$ has a «largest» element (τ_M), the torsion functor *associated* with M.

EXAMPLE. Let R be a commutative ring, S a multiplicatively closed subset of R, then S defines a torsion functor μ_S, where

$$\mu_S(N)=\{n\in N|sn=0 \text{ for some } s\in S\}.$$

If P is a prime ideal of the (commutative) ring R, and $S=R\setminus P$, then S is multiplicatively closed in R, and we write μ_P in stead of $\mu_{R\setminus P}=\mu_S$

2.2. Let P be a proper prime ideal of the commutative ring R; then

$$\tau_{R/P}=\mu_P \text{ (where } \mu_P \text{ means } \mu_S \text{ or } \mu_{R\setminus P}).$$

Pr.: If $a\in R\setminus P$, then $ax\in P\to x\in P$, hence $\mu_{R\setminus P}(R/P)=0$, since $sa'=0(a'\in R/P, s\in R/P)$ implies that $s=0$. That implies, that $\mu_{R\setminus P}\leqslant\tau_{R/P}$. Now let $\boldsymbol{M}$ be an ideal $\boldsymbol{M}\notin\boldsymbol{L}_{\mu_{R\setminus P}}$. Then $\boldsymbol{M}$ contains no element of $R\setminus P$, *i.e.* $\boldsymbol{M}\subset P$, and then $\boldsymbol{M}$ annihilates all elements of R/P, *i.e.* $\boldsymbol{M}\nsubseteq\boldsymbol{L}_{\tau_{R/P}}$ and thus $\tau_{R/P}=\mu_P$.

2.3. If σ is a hereditary torsion functor of R-Mod, then there exists an R-module S such that $\sigma=\tau_S$; (see [3]).

(I.e. every hereditary torsion functor σ is a τ_S for a suitable $S\in R$-Mod).

Led X be a subsect of all hereditary torsion functors, M an R-module; define $\sigma(M)=\cap\rho(M)$; then ρ is a hereditary torsion functor and $\sigma(M)\subseteq\rho(M)$ for all $\rho\in X$ and all $M\in R$-Mod. Furthermore we have: if $\tau<\rho$ for all $\rho\in X$, then $\rho\leqslant\sigma$. Thus it is reasonable to call $\sigma=\inf(X)$, and $\sigma(M)=\bigcap_{\rho\in X}\rho(M)$ the $\inf(\rho(M))$, $\rho\in X$.

2.4. Let $M=\bigoplus_\alpha M_\alpha$, and $\rho_\alpha=\tau_{M_\alpha}$ $(\alpha\in A)$; then $\tau_M=\inf_{\alpha\in A}(\tau_{M_\alpha})$.

Pr.: Since $M_\alpha\subset M$, we have $\tau_M<\tau_{M_\alpha}$ $(\forall\alpha\in A)$, *i.e.* $\tau_M\leqslant\bigcap_\alpha\tau_{M_\alpha}$.

If $\sigma<\tau_M(\forall\alpha)$, then $\sigma<\tau_M$, and that implies that $\tau_M=\inf_{\alpha\in A}(\tau_{M_\alpha})$.

Moreover it implies that $\tau_M(N) = \bigcap_{\alpha \in A} \tau_{M_\alpha}(N)(\forall N \in R-\text{Mod})$.

Let $M \neq 0$ be a uniform R-module, and suppose that we have for the associated torsion functor τ_M, $\tau_M = \rho \wedge \sigma$ for some pair of torsion functors ρ, σ. Then $0 = = \tau_M(M) = \rho(M) \cap \sigma(M)$. Since M is uniform we have $\rho(M) = 0$ or $\sigma(M) = 0$, and that implies that $\rho \leq \tau_M$ or $\sigma \leq \tau_M$; but from $\tau_M \leq \rho$ (resp. $\tau_M \leq \sigma$) we conclude that $\tau_M = \rho$ or $\tau_M = \sigma$; *i.e.*

2.5. If $M \neq 0$ is a uniform R-module, then τ_M is indecomposable in the sense that $\tau_M = \rho \wedge \sigma \rightarrow \tau_M = \rho$ or $\tau_M = 0$.

If M is an injective uniform R-module, then M is an indecomposable injective R-module. Now suppose that R is a (left) Noetherian ring; then every injective R-module is a direct sum of indecomposable injective submodules. If now τ is a hereditary torsion functor, then (by 2.3) there exists an R-module M such that $\tau = \tau_M = \tau_{\hat{M}}$.

Since $\hat{M} = \bigoplus_\alpha M_\alpha$ is a direct sum of indecomposable injective sumbodules M_α, we have

$$(**) \qquad \tau = \inf_{\alpha \in A}(\tau_{M_\alpha}),$$

and each τ_{M_α} is an indecomposable torsion functor in the sense of 2.5.:

2.6. If R is a (left) Noetherian ring, every hereditary torsion functor τ of R-Mod is generated by all τ_{M_α}, where M_α is an indecomposable injective R-module, and where «generated» has the sense of (**).

We return to the hereditary torsion functor τ_K associated with a fixed R-module K, and

$$\tau_K(M) = \bigcap \{\ker(\phi) | \phi: M \rightarrow \hat{K}\}.$$

Then $\tau_K(K) = 0$, $\tau_K \in \Lambda(K)$, while $\rho \in \Lambda(K)$ iff $\rho \leq \tau_K$.

For any R-module M we then have, if $\rho \in \Lambda(K)$:

$\rho(M) \subseteq \tau_K(M)(\forall M \in R\text{-Mod})$, $\rho(M/\tau_K(M)) \subseteq \tau_K(M/\tau_K(M)) = 0$ for all $\rho \in \Lambda(K)$ and all $M \in R$-Mod, *i.e.* $\rho(M/\tau(M)) = 0$. Conclusion:

2.7. If $K \neq 0$ is a fixed R-module, $\rho \in \Lambda(K)$, and $M \in R$-Mod, then $\tau_K(M)$ is, among the submodules $\rho(M)$, the unique maximal submodule $N \subseteq M$ with $\rho(M/N) = 0$. We have $\tau_K(M) = M$ if and only if $\text{Hom}_R(M; \hat{K}) = 0$.

2.8. If K_1 and K_2 are complements of a submodule $K \subseteq M$, then $\tau_{K_1} = \tau_{K_2}$.

Pr.: If $\pi: M \rightarrow M/K$, then $K_i \cong \pi(K_i) \subseteq M/K (i = 1, 2)$; therefore $\tau_{K_1} = \tau_{K_2} = \tau_{M/K}$.

If therefore $K \neq 0$ is a submodule of the uniform module M, then $\tau_K = \tau_M$.

In the following we consider a locally uniform R-module M, *i.e.* every submodule $0 \neq N \subseteq M$ contains a uniform submodule. Let $\{N_i | i \in I\}$ be a maximal independent set of uniform submodules $\neq 0$ of M, *i.e.* $\Sigma N_i = \bigoplus_i N_i \subseteq_e M$.

For each N_{i_0} $(i_0 \in I)$ we choose a complement $N_{i_0}^c \supseteq i \bigoplus_{i \neq i_0} N_i$, then $\bigcap_i N_i^c = 0$

and this is an irredundant intersection of essentially closed submodules of M. Identifying $M_i = M/N_i^c$, we have $N_i \subseteq_e M_i = M/N_i^c$, thus $M \cong \underset{i\in I}{\underset{\sim}{x}} M_i = \underset{i\in I}{\underset{\sim}{x}} M/N_i^c$.

According to the associated torsion functor τ_M of the locally uniform R-module M we note that $N = \bigoplus_{i\in I} N_i \subseteq_e M$, *i.e.* $\tau_M = \tau_N = \inf_{i\in I}(\tau_{N_i})$.

On the other hand $M = \underset{i\in I}{\underset{\sim}{x}} M_i$ is an essential subdirect product in the sense that $M \cap M_i \subseteq_e M_i\,(i \in I)$. That implies, that $\tau_{M\cap M_i} = \tau_{M_i}$; since $M \cap M_i \subseteq M$, we have $\tau_M \subseteq \tau_{M\cap M_i} = \tau_{M_i}\,(\forall i \in I)$. We prove that if $\sigma < \tau_{m_i}\,(\forall i \in I)$, then $\sigma < \tau_M$. If $\sigma < \tau_{M_i}$, then $\sigma(M) \subset \tau_{M_i}(M) = \bigcap \{\ker \phi \mid \phi: M \to \hat{M}_i\} \subseteq N_i^c$, therefore

$$\pi_i\sigma(M) \subseteq \pi_i\tau_{M_i}(M) \subseteq \pi_i N_i^c = 0, \quad i.e. \quad \sigma(M) \subseteq \bigcap_i N_i^c = 0,\ \sigma \leqslant \tau_M\,,$$

and therefore

$$\tau_M = \inf_{i\in I}(\tau_{M_i}).$$

Summarizing we have:

2.9. If is a locally uniform R-module, $\{N_i \mid i \in I\}$ a maximal independent set of non-zero uniform submodules of M, $\{N_i^c \mid i \in I\}$ a set of complements of the N_i in M, $N_{i_0}^c \supseteq i \bigoplus_{i\neq i_0} N_i$, then we have:

(i) $\bigoplus_{i\in I} N_i \subseteq_e M$;

(ii) $M \cong \underset{i}{\underset{\sim}{x}} M/N_i^c$ is an irredundant essential subdirect product of the uniform modules $M_i = M/N_i^c\,(i \in I)$;

(iii) for the corresponding associated indempotent torsion functors τ_M, τ_{N_i}, τ_{M_i} we have the relations: $\tau_M = \inf_{i\in I}(\tau_{M_i}) = \inf_{i\in I}(\tau_{N_i})$.

The associated torsion functor τ_M of M represented in 2.9 (iii) is independent of the representation (ii) of M. Therefore we define: if an essential submodule N of M is isomorphic with an essential submodule N' of the R-module M', then M and M', are called *equivalent*, and it follows that $\tau_M = \tau_{M'}$. Using the notations of 2.9 we suppose that $\{N_i \mid \mid i \in I\}$ and $\{L_j \mid j \in J\}$ are two maximal independent sets of uniform submodules of M. Then there exists a $1-1-$ map $\phi: I \to J$ such that N_i and $L_{\phi(i)}$ are equivalent $(\forall i \in I)$.

The corresponding representations of M are

$$M \cong \underset{i\in I}{\underset{\sim}{x}} M/N_i^c \quad \text{and} \quad M \cong \underset{j\in J}{\underset{\sim}{x}} M/L_j^c.$$

Since $N_i \subseteq_e M/N_i^c$, $L_j \subseteq_e M/L_j^c$, the equivalence of N_i and $L_{\phi(i)}$ implies that the representation 2.9 (ii) of M as a subdirect product of uniform modules is unique up to equivalence of the components. From the equivalence of N_i and $L_{\phi(i)}$ it follows moreover that $\tau_{N_i} = \tau_{L_{\phi(i)}}(\forall i)$. Therefore:

2.10. If M is a locally uniform R-module, then the associated torsion functor τ_M, as expressed in 2.9 (iii) is independent of the choice of a maximal independent set of non-zero uniform submodules of M.

References

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