Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Aldo Bressan

On control theory and its applications to certain problems for Lagrangian systems. On hyper-impulsive motions for these. III. Strengthening of the characterizations performed in parts I and II, for Lagrangian systems. An invariance property.

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **82** (1988), n.3, p. 461–471.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1988_8_82_3_461_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1988.

Atti Acc. Lincei Rend. fis.
(8), LXXXII (1988), pp. 461-471

Fisica matematica. — *On control theory and its applications to certain problems for Lagrangian systems. On hyper-impulsive motions for these. III. Strengthening of the characterizations performed in parts I and II, for Lagrangian systems. An invariance property.*(*) Nota (**) del Corrisp. ALDO BRESSAN.

ABSTRACT. — In [1] I and II various equivalence theorems are proved; e.g. an ODE $(\mathcal{E})\ \dot{z} = F(t, z, u, \dot{u})\ (\in \mathbb{R}^m)$ with a scalar control $u = u(\cdot)$ is linear w.r.t. $\dot{u}$ iff (α) its solution $z(u, \cdot)$ with given initial conditions (chosen arbitrarily) is continuous w.r.t. u in a certain sense, or iff (β) $z(u, \cdot)$ satisfies certain conditions by which 1^{st}-order discontinuities of u and $\dot{u}$ can be treated satisfactorily.

In the case when, for $z = (q, p)$ equation $(\mathcal{E})$ is a semi-Hamiltonian system — equivalent to a system of Lagrangian equations of a general type —, the importance or compulsory character in many situations, of the conditions hinted at in (α) and (β), have received some intuitive justifications in [1] II. In the present paper some of these are replaced by theorems and thus the importance of the above linearity is strengthened. E.g. this linearity is shown, roughly speaking, to follow from the continuity (in the afore-mentioned sense) of the function $u \vdash q(u, \cdot)$ alone.

In the above semi-Hamiltonian case, the linearity of equation $(\mathcal{E})$ w.r.t. u is also proved to be invariant under certain transformations of Lagrangian co-ordinates.

KEY WORDS: Lagrangian systems; Feed back theory.

RIASSUNTO. — *Sulla teoria dei controlli e le sue applicazioni a certi problemi per sistemi Lagrangiani. III. Rafforzamento delle caratterizzazioni effettuate nelle parti I e II, per sistemi Lagrangiani. Una proprietà d'invarianza.* In [1] I and II si dimostrano vari teoremi di equivalenza; per es., un'equazione differenziale ordinaria $(\mathcal{E})\ \dot{z} = F(t, z, u, \dot{u})\ (\in \mathbb{R}^m)$ contenente un controllo scalare $u = u(\cdot)$, è lineare in $\dot{u}$ se e solo se (α) la soluzione $z(u, \cdot)$ di $(\mathcal{E})$ con date condizioni iniziali (scelte ad arbitrio) è continua rispetto ad u in un certo senso, oppure se e solo se (β) $z(u, \cdot)$ verifica certe condizioni che permettono di trattare soddisfacentemente i casi di discontinuità di prima specie per u e $\dot{u}$.

Nel caso che per $z = (q, p)$ la $(\mathcal{E})$ sia un sistema semi-Hamiltoniano — equivalente ad un sistema di equazioni di Lagrange di tipo generale —, l'importanza e magari l'irrinunciabilità in molte situazioni, delle condizioni accennate in (α) e (β) è motivata, in [1] II, con considerazioni intuitive. Nel presente lavoro, alcune di esse sono sostituite con teoremi rafforzando così l'importanza della suaccennata linearità. Per es., grosso modo, si dimostra che questa segue dalla continuità della sola $u \vdash q(u, \cdot)$ nel senso suaccennato.

Nel caso semi-Hamiltoniano suddetto, si dimostra pure un teorema di invarianza della linearità della $(\mathcal{E})$ in $\dot{u}$, rispetto a certe trasformazioni di co-ordinate Lagrangiane.

(*) The present work has been performed within the activity sphere of group 3 of the Consiglio Nazionale delle Ricerche, in the year 1987.
(**) Presentata nella seduta del 20 novembre 1987.

N 12 Introduction to Part III.

The linearity w.r.t. (with respect to) $\dot{u}$ of the ODE $(1.1)_1$ with the scalar control u is shown in Part. I — i.e. [1] I — to be equivalent to e.g. the C^0-controllability of the (functional) parameter u in $(1.1)_1$, which roughly speaking means that the (local) C^1-solution $z(u,\cdot)$ of Cauchy problem (1.1) for any admissible C^1-choice of $u = u(\cdot)$ is a continuous function of u, when the (appropriate) sup norms $\|\cdot\|_0$ are used for both $u(\cdot)$ and $z(u, \cdot)$. This continuity condition is generally regarded as important and even compulsory in many situations, which implies the importance of the linearity of equation $(1.1)_1$ w.r.t. $\dot{u}$.

Results such as the above equivalence are applied in Part II to the SHE (i.e. semi-Hamiltonian equation) (11.6), which describes dynamically the Lagrangian system $\Sigma_{\hat{\gamma}}$ obtained from Σ by identifying its M-tuple γ of co-ordinates (in $\chi = (q, \gamma)$ with any (admissible) control $\hat{\gamma}(t) = \tilde{\gamma}[u(t)]$, where $\tilde{\gamma}(\cdot) \in C^3$ while $u(\cdot) \in C^1$, and by implementing the kinematic relation $\gamma = \hat{\gamma}(t)$ as the addition of M frictionless constraints. By (11.2-4)

$$(12.1) \quad p_h = p_h(t,q,\dot{q},\gamma,\dot{\gamma}) \equiv \partial T/\partial \dot{q}^h = \sum_{k=1}^{N} a_{hk}\dot{q}^k + B_h + \sum_{\varrho=1}^{M} A_{h,N+r} \quad (\gamma = \hat{\gamma}(t) = \tilde{\gamma}[u(t)])$$

so that, at least at first sight, it may appear excessive to require the whole afore-mentioned continuity of $z(u,\cdot) = (q(u,\cdot),\ p(u,\cdot))$ w.r.t. $u = u(\cdot)$. Only the one of $q(u,\cdot)$ may seem reasonable. E.g. in case $\|u_a(\cdot)\|_0 \to 0$ while $\|\dot{u}_a(\cdot)\|_0$ keeps $\geqslant 1$ for $a \to 0^+$, it seems natural to expect that condition (i) $\|\dot{q}(u_a,\cdot) - \dot{q}(0,\cdot)\|_0 \to 0$ should be violated — which generally occurs — and that, consequently, by (12.1) condition (ii) $\|p(u_a(\cdot),\cdot) - p(0,\cdot)\|_0 \to 0$ might also be violated. Therefore some intuitive reasons for requiring the above continuity for $u \vdash p(u,\cdot)$ are given in Part II — see ftn. 2 in N8 and ftn. 3 in N9.

In the present paper, e.g., those reasons are replaced by rigorous reasonings in that Theor. 14.1 asserts, roughly speaking, a result slightly stronger than the following one: the above continuity of $u \vdash q(u,\cdot)$ — i.e. the C^0-semicontrollability of the parameter u in the SHE (11.6) with $\gamma = \tilde{\gamma}(u)$ — is sufficient for the linearity w.r.t. $\dot{u}$ of the same equation. Among other things this implies the above continuity for $u \vdash p(u, \cdot)$, so that the importance of e.g. the afore-mentioned equivalence result obtained in Part II, is strengthened. Here other results of Part II are strengthened in the same way. Among them let us mention those on BVC^0-controllability and 1-dimensional BVC^0- or C^0-controllability, and especially Theors. 9.1 and 10.2 on fitness for jumps and 1-dimensional fitness for jumps respectively. For instance here the condition of 1-dimensional semifitness for jumps of the paremeter γ in the SHE (11.6), which is weaker than the one of 1-dimensional fitness for jumps, is proved to imply this fitness via the linearity of (11.6) w.r.t. $\dot{\gamma}$.

Note that the result above is based on Theor. 15.1, which belongs to the theory of hyper-impulsive motions and takes a suitable requirement of continuity for jumps into account; and therefore its proof has some features in common with the one of Theor. 14.1, which belongs to control theory; however the two proofs have this important difference. The second involves a family $u_a(\cdot)$ of oscillating C^1-controls, which tend in the sup norm to a linear control, while the former involves a family $u_{rj}(\cdot)$ of continuous controls which are non-oscillating, (only) piecewise C^1, capable of simulating a jump j, and tending to a constant control when $j \to 0$.

In connection with the SHE (11.6), it is natural to prove the invariance of C^0-semicontrollability under certain transformations — see (13.2) — of the Lagrangian co-ordinates of Σ [Theor. 13.1]. This implies — see Corollary 14.4 — the invariance under the same transformations of the linearity of (11.6) w.r.t. $\dot{\gamma}$. (1).

N 13 C^0- *and* BVC^0-*semicontrollability for semi-Hamiltonian equation, their invariance under a certain group* G *of coordinate transformations.*

Let us consider the (2N-dimensional) SHE (i.e semi-Hamiltonian equation) (11.6) with $\gamma = \tilde{\gamma}(u)$ — see (8.2) —, for any (admissible) choice of $\tilde{\gamma}(\cdot) \in C^3$. It is not restrictive to identify it with $(2.2)_1$ for $z = (q, p)$, so that $m = 2N$; hence it is equivalent to $(2.3)_1$ for $x = (t, u, q, p)$, whence $n = 2 + 2N$ — see (2.1).

Furthermore consider Cauchy problem (2.2) for $\bar{z} = (\bar{q}, \bar{p})$. By (2.4) it is equivalent to problem (2.3) for $\bar{x} = (\bar{t}, \bar{u}, \bar{p}, \bar{q})$ where, for some compact segments Δ and Δ' in $\mathbb{R}$, $\bar{t} \in \mathring{\Delta}$, $u = u(\cdot) \in C^1(\Delta, \Delta')$, and $\bar{u} = u(\bar{t})$. The C^1-solution $x(u, \cdot)$ of (2.3) in Δ, if it exists, has the form

$$(13.1) \qquad x(t) \equiv x(u,t) \equiv (x_1(t), \ldots, x_n(t)) \equiv (t, u(t), q'(t), p(t)).$$

Hence it contains the C^1-solution $z(u, \cdot) = (q(u, \cdot), p(u, \cdot)) = (q(\cdot), p(\cdot))$ in Δ of problem (2.2), and it is also determined by $z(u, \cdot)$.

Def. 13.1 [13.2] - *Assume that the* ODE $(2.2)_1$, *with* $z = (q, p)$ *is the SHE* (11.6) *with* $\gamma = \tilde{\gamma}(u)$ (*as above*). *Then I say that the* (*functional scalar*) *parameter* u *in it is* C^0-[BVC^0-]*semicontrollable, if* [*if for some* $b > 0$,] *for all* $\bar{t}$, $\bar{u}$, $\bar{z}$, $\bar{v}$, Δ, Δ', *and* $\mathcal{U}$, *conditions* (a_1) *to* (a_2) *in Def.* 2.1 [*and* (a_3) *in Def.* 2.2] *imply that*

($\mathfrak{A}$) $q(u, \cdot)$ *is a continuous function of* $u = u(\cdot)$ *when the sup norm* $\|\cdot\|_0$ *is used on both* $C^1(\Delta, \Delta')$ *for* u *and* $C^1(\Delta, \mathbb{R}^N)$ *for* $q(u, \cdot)$ (2).

Now remember that equations (11.6) for the Lagrangian system $\Sigma \hat{\gamma}$ are expressed in terms of Σ's hinetic energy - see (11.1) - and the components $\mathcal{Q}_{\mathfrak{R}}$ of the applied forces for the Lagrangian system Σ, referred to the (system of) $\mathfrak{N} = N + M$ co-ordinates $\chi = (q, \gamma) = (q^1, \ldots, q^N, \gamma^1, \ldots, \gamma^M)$.

Def. 13.3 [13.4] - *I shall say that the* (*M-dimensional functional*) *parameter* γ *in the* SHE (11.6) *is* 1-*dimensionally* C^0-[BVC^0]-*semicontrollable if, for every* (*admissible*) C^3-*path* $\tilde{\gamma} : \Omega \to \mathbb{R}^M$ *where* $\Omega (\subset \mathbb{R})$ *is a bounded segment, the substitution* $\gamma = \tilde{\gamma}(u)$ (*and*

(1) Among other things, Corollary 14.3 allow the author to simplify considerably his original proof of one among the main theorems in [2] — see ftn. 4.

(2) Strictly speaking, Def. 13.1 [13.2] defines *weak* C^0- [BVC^0-]*semicontrollability* — see ftns. 2 and 8 in Part. I. The analogue holds for Def. 13.3 [13.4].

$\dot{\gamma} = \tilde{\gamma}'(u)\dot{u}$) *renders* (11.6) *an equation of the form* $(2.2)_1$ *in which the* (*functional*) *parameter* u *is* C^0-[BVC^0-] *semicontrollable — see* ftn. 2.

Note that Defs. 2.1-3 imply assertions (α) and (β) below.

(α) C^0-*semicontrollability implies* BVC^0-*semicontrollability; and the former* [*latter*] *follows from* C^0-[BVC^0-] *controllability.*

(β) *The* 1–*dimensional* C^0-*semicontrollability of the parameter* γ *in the* SHE (11.6) *implies* γ*'s* 1-*dimensional* BVC^0-*semicontrollability*; *and the former* [*latter*] *follows from* γ*'s* 1-*dimensional* C^0-[BVC^0-] *controllability.*

Now consider the C^3-transformation $\chi^{\mathcal{R}} = \chi^{\mathcal{R}}(\chi^*, t)$ ($\mathcal{R} = 1, \ldots, \mathfrak{N}$) of $\chi = (q, \gamma)$ into the system $\chi^* = (q^*, \gamma^*)$ of Lagrangian co-ordinates for Σ, which has the form given by the first two of the equalities

$$q = q(q^*, \gamma, t), \quad \gamma = \gamma(\gamma^*); \quad q^* = q^*(q, \gamma, t), \quad \gamma^* = \gamma^*(\gamma) \tag{13.2}$$

and is invertible by means of $(13.2)_{3\text{-}4}$. Such transformations form a group G.

Theor. 13.1 - *Consider the* $(\chi \to \chi^*)$-*transform* $(11.6)^*$ *of the SHE* (11.6) — *see* (13.2). *The* 1-*dimensional* C^0-*semicontrollability of the functional parameter* γ *in* (11.8) *is equivalent to the one of* γ^* *in* $(11.6)^*$ (3).

Indeed any C^3-path $\tilde{\gamma}(\cdot): \Omega \to \mathbb{R}$ for (11.6) is transformed by $(13.2)_4$ into a C^3-path $\tilde{\gamma}^*(\cdot): \Omega \to \mathbb{R}$ for $(11.6)^*$.

Now identify (11.6) with $(7.1)_1$ where $z = (q, p)$, and $(11.6)^*$ with $(13.3)_1$ below.

$$\dot{z}^* = \Phi(t, \gamma^*, z^*, \dot{\gamma}^*), \quad z^*(\bar{t}) = \bar{z}^* = (\bar{q}^*, \bar{p}^*) \quad (z^*(\cdot) = (q^*(\cdot), p^*(\cdot))). \tag{13.3}$$

By regarding $\gamma = \tilde{\gamma}(\cdot) = \tilde{\gamma}[u(\cdot)]$ [$\gamma^* = \hat{\gamma}^*(\cdot) = \tilde{\gamma}^*[u(\cdot)]$], problem, (7.1) [$(13.3)_{1\text{-}2}$] is transformed by $(13.2)_{1\text{-}2}$ into problem $(13.4)_{1\text{-}2}$ [$(13.4)_{3\text{-}4}$] below.

$$\begin{aligned} &\dot{z} = \Phi[t, \tilde{\gamma}(u), z, \tilde{\gamma}'(u)\dot{u}], \quad z(\bar{t}) = \bar{z}; \\ &\dot{z}^* = \Phi[t, \tilde{\gamma}^*(u), z^*, \tilde{\gamma}^{*\prime}(u)\dot{u}], \quad z^*(\bar{t}) = \bar{z}^*. \end{aligned} \tag{13.4}$$

As far as the $(\chi \to \chi^*)$-transform $p^*(\cdot)$ of $p(\cdot)$ is concerned, let us note that, since $(13.4)_1$ is (11.6) for $\gamma = \tilde{\gamma}(\cdot)$ and $(11.6)_1$ is equivalent to $(11.4)_1$, $p(\cdot)$ is thus expressed in terms of $q(\cdot)$, $\dot{q}(\cdot)$, $u(\cdot)$, and $\dot{u}(\cdot)$. Likewise — $(13.4)_3$ being, say, $(11.6)^*$ for $\gamma^* = \tilde{\gamma}^*(\cdot)$ — relation $(11.4)_1^*$ for $\gamma^* = \tilde{\gamma}^*(\cdot)$ expresses $p^*(\cdot)$ in terms of $q^*(\cdot)\,(= q^*(u, \cdot))$, $\dot{q}^*(\cdot)$, $u(\cdot)$, and $\dot{u}(\cdot)$. Furthermore, by (13.2),

$$q^*(u, t) \equiv q^*[q(u, t), \tilde{\gamma}(u(t)), t], \quad \tilde{\gamma}^*(u) \equiv \gamma^*[\tilde{\gamma}(u)]. \tag{13.5}$$

By the considerations above, given problem $(13.4)_{1\text{-}2}$, its $(\chi \to \chi^*)$-transform $(13.4)_{3\text{-}4}$ — and in particular $\bar{z}^* = (\bar{q}^*, \bar{p}^*)$ — is completely determined. Furthermore note that from $(11.4)_1^*$ for $\gamma^* = \tilde{\gamma}^*(u)$, we only see that $p^*(\cdot)$ is continuous; however we can

(3) The analogues of Theor. 13.1 for BVC^0-semicontrollability or C^0-, BVC^0, or $\mathcal{L}^1$-controllability hold. But it is convenient to derive them as straightforward consequences of Corollary 14.2. This holds especially for the last three analogues.

assert that this $p^*(\cdot)$ is in C^1 because if we introduced it as a part of the solution $z^*(u,\cdot) = (q^*(\cdot), p^*(\cdot))$ of problem $(13.4)_{3\text{-}4}$, since $(13.4)_3$ is $(11.6)^*$, $(11.4)_1^*$ would hold again and $z^*(u,\cdot) \in C^1$.

At this point it is clear that by our assumptions, which are symmetric with respect to (11.6) and $(11.6)^*$, the entities $\bar{t}$, $\bar{u}$, $\bar{x}$, $\bar{v}$, Δ, Δ', and $\mathcal{U}$ satisfy condition (a_1) $[(a_2)]$ in Def. 2.1 (related to problem $(13.4)_{1\text{-}2}$) iff $\bar{t}$, $\bar{u}$, $\bar{z}^*$, $\bar{v}$, Δ, Δ', and $\mathcal{U}$ satisfy its analogue, say (a_1^*) $[(a_2^*)]$ for problem $(13.4)_{3\text{-}4}$.

Now assume the 1-dimensional C^0-semicontrollability of the parameter γ in (11.6). In order to deduce the one of the parameter γ^* in $(11.6)^*$, choose the C^3-path $\tilde{\gamma}:\Omega \to \mathbb{R}^M$ (where Ω is a bounded segment) arbitrarily; and regard $(13.4)_3$ $[(13.4)_1]$ as obtained from $(11.6)^*$ [(11.6)] by replacing γ^* $[\gamma]$ with $\gamma^*(\cdot)$ $[\gamma(\gamma^*(\cdot))]$.

Assume that $\bar{t}$, $\bar{u}$, $\bar{z}^*$, ..., $\mathcal{U}$ satisfy conditions (a_1^*) and (a_2^*), so that $\bar{t}$, $\bar{u}$, $\bar{z}$, ..., $\mathcal{U}$ satisfy conditions (a_1) and (a_2) in Def. 2.1. Then the above 1-dimensional C^0-semicontrollability assumption implies, by Def. 13.3, the C^0-semicontrollability of the parameter u in the ODE $(13.4)_1$, so that by Def. 13.1 condition $(\mathfrak{A})$ there holds.

Assume that $u, u_1, u_2, \ldots \in \mathcal{U}$ and (i) $\|u_i - u\|_0 \to 0$, so that (ii) $\|q(u_i,\cdot) - q(u,\cdot)\|_0 \to 0$ by $(\mathfrak{A})$. Since the transformation (13.2) is C^3, (i), (ii), and $(13.5)_1$, imply that $\|q^*(u_i,\cdot) - q^*(u,\cdot)\|_0 \to 0$. Hence by the arbitrariness of u, u_1, u_2, ..., $q^*(u,\cdot)$ satisfies condition $(\mathfrak{A})$ in $q(u,\cdot)$. Then, by Def. 13.1, the parameter u is C^0-semicontrollable in $(13.4)_3$. By the arbitrariness of $\tilde{\gamma}^*(\cdot)$, the parameter γ^* is 1-dimensionally semicontrollable in $(16.6)^*$. Thus a half of the theorem is proved. the remaining half holds by symmetry.

q.e.d.

N 14 – *Equivalence of the* 1*-dimensional* BVC^0*-semicontrollability of the parameter* γ *in the* SHE (11.6) *with e.g. its* BVC^0*-controllability, via the linearity in* $\dot{\gamma}(= \tilde{\gamma}(u)\dot{u})$ *of* (11.6). *An invariance property of this linearity.*

Theor. 14.1 – *Assume that* (i) *the* ODE $(2.2)_1$ *with* $z = (q, p)$ *is the* SHE (11.6) *with* $\gamma = \tilde{\gamma}(u)$ (*and* $\dot{\gamma} = \tilde{\gamma}'(u)\dot{u}$ *as in* N 13) *and* (ii) *the parameter* u *in it is* BVC^0*-semicontrollable. Then* $(2.2)_1$ *is at most linear in* $\dot{u}$.

Proof. Remember the proof of Theor. 3.1 — which is the analogue of Theor. 14.1 for BVC^0-controllability —, where the assumption of u's BVC^0-controllability is mentioned only below (3.8). In that proof the version (2.3) of (2.2) is referred to; and as an hypothesis for reduction ad absurdum it is assumed that (a) $f_{vv}(\bar{x}, \bar{v}) \neq 0$ *for some* $(\bar{x}, \bar{v}) \in \mathcal{V}$, *the domain* $\mathfrak{D}_f$ *of* f (*where* $x = (t, u, z)$).

An $r_0 > 0$ with $\bar{B}((\bar{x}, \bar{v}), r_0) \subset \mathcal{V}$ is considered, and for the sake of simplicity it is assumed that $\bar{t} = 0$. The control $(u_{\epsilon,a}(\cdot) =) u_\epsilon(\cdot)$ is defined by $(3.3)_1$ for $\epsilon \geq 0$ and all $a \in (0, r_0)$, and (b) *its limit for* $\epsilon \to 0^+$ *is noted to be* $(u_{0a}(t) \equiv) u_0(t) \equiv \bar{u} + \bar{v}t$. The solution $y_\epsilon(\cdot)$ (on $\mathbb{R}$) — see (3.5) — of Cauchy problem (3.4) and the maximal solution $x_\epsilon(\cdot)$ of Cauchy problem (2.3) with $u = u_\epsilon(\cdot)$ are also considered for $\epsilon \geqslant 0$. Furthermore inequalities (3.8) are deduced.

At this point, to prove Theor, 14.1, note that by the assumed BVC^0-

semicontrollability of the parameter u in $(2.3)_1$, — see Def. 13.2 —, there is some $b > 0$ such that (c) *for all* $\bar{x}$, Δ, Δ', *and* $\mathcal{U}$, *conditions* (a_1) *to* (a_3) *in Def.* 2.1-2 *imply condition* $(\mathfrak{A})$ in *Def.* 13.2.

Below (3.8) the analogue of (c) for condition (A) in Def. 2.1 is asserted; but all deductions after this analogue and before the sentence including (3.13), are also valid on the basis of the present hypotheses.

Thus, for any $\eta \in (0, |\gamma(\bar{x})|)$ — see (3.1) and ftn. 3 in [1] I — there are some $r_1 \in (0, r_0) \cap (0, b)$, $r_2 \in (0, r_1)$, $\tau \in (0, r_2/\eta)$, and $\epsilon_0 \gneqq 0$, such that

(d) *for* $a = r_1$ *and* $0 \leq \epsilon \leq \epsilon_0$ *we have that* $[0, \tau] \subseteq \mathfrak{D}_{x\epsilon(\cdot)} \cap \mathfrak{D}_{x_0(\cdot)}$;

and — see (3.12) and below (3.11) and (3.12) —

$$x_\epsilon(t), y_\epsilon(t) \in B(\bar{x}, r_2/2) \text{ for } t \in [0, \tau], 0 \leq \epsilon \leq \epsilon_0, \text{ and } a = r_1. \tag{14.1}$$

From now on let $x_\epsilon(\cdot)\,[y_\epsilon(\cdot)]$ denote the restriction to $[0, \tau]$ of the function denoted by $x_\epsilon(\cdot)[y_\epsilon(\cdot)]$ so far; and set $\mathcal{U} = \{u\epsilon(\cdot)\}_{0 \leq \epsilon \leq \epsilon_0}$ again. Thus conditions (a_1) to (a_3) in Defs. 2.1-2 hold again (for $a = r_1$). Hence, by Def. 13.2, the BVC^0-semicontrollability of the parameter u in $(2.3)_1$ implies the first of the relations

$$\lim_{\epsilon \to 0^+} \|q_\epsilon(\cdot) - q_0(\cdot)\|_0 = 0, \quad \lim_{\epsilon \to 0^+} \|x_\epsilon(\cdot) - x_0(\cdot)\|_0 = 0, \quad (x = (t, u, q, p)), \tag{14.2}$$

while the assumptions in Theor. 3.1 imply (3.13), i.e. $(14.2)_2$. The deduction of (3.14-15) is independent of $(14.2)_1$, and it is valid here; the same holds for the deduction of the assertion involving $(3.15)_2$, which obviously implies that

(e) *there is an* $\epsilon_1 \in (0, \epsilon_0)$ *such that, for* $a = r_1$ *and* $\tau_0 = \tau/2$

$$|x_\epsilon(t) - x_0(t)| > (a^2\,|\gamma(\bar{x})| - \eta)\,t > 0 \quad \forall t \in [\tau_0, \tau] \quad \forall \epsilon \in (0, \epsilon_1). \tag{14.3}$$

Hence $(14.2)_2$ is false. Furthermore, since $(14.2)_2$ follows from the condition

$$\lim_{\epsilon \to 0^+} |x_\epsilon(\tau) - x_0(\tau)| = 0, \tag{14.4}$$

this is also false. Setting (f) $x_\epsilon(\cdot) = (\overset{1}{x}_\epsilon(\cdot), \overset{2}{x}_\epsilon(\cdot), q_\epsilon(\cdot), p_\epsilon(\cdot))$ $(0 \leqslant \epsilon \leqslant \epsilon_1)$, (2.4) yields (g) $\overset{1}{x}_\epsilon(t) \equiv t$ and $\overset{2}{x}_\epsilon(t) \equiv u_\epsilon(t) \,\forall \epsilon \in [0, \epsilon_1]$. Furthermore, by (b), (h) $\|u_\epsilon(\cdot) - u_0(\cdot)\|_0 \to 0$. Hence by $(14.2)_1$ the falsity of (14.4) implies the one of the relation $|p_\epsilon(\tau) - p_0(\tau)| \to 0$. Then, for some $\varrho \in (0, r_2/2)$ and some (strictly) decreasing sequence $\epsilon_1, \epsilon_2, \ldots$ $(\epsilon_1 < \epsilon_0)$ the first two of the relations

$$\epsilon_i \to 0, \quad \varrho \leqslant |p_{\epsilon_i}(\tau) - p_0(\tau)| \leqslant r_2/2 \tag{14.5}$$

hold. The third follows from (14.1). By $(14.5)_{2\text{-}3}$ $p_{\epsilon_i}(\tau) \in K = {}_D\bar{B}(p_0(\tau), r_0/2) - \mathring{B}(p_0, \varrho)$, a compact set. Hence for some $p' \in K$ and some strictly increasing sequence $r \vdash i_r, p_{(r)} \to p'$, *where* $p_{(r)}(\cdot) = p_{\epsilon_i}(\cdot)$ *for* $i = i_r (r \in \mathbb{N}^*)$. Then, by $(14.2)_1$ and (f) to (h)

$$x_{(r)}(\tau) \to (\tau, u_0(\tau), q_0(\tau), p') \text{ where } x_{(r)}(\cdot) = x_\epsilon(\cdot) \text{ for } \epsilon = \epsilon_{i_r} \,(r \in \mathbb{N}^*). \tag{14.6}$$

Then

(14.7) $$\lim_{r\to\infty} \|x_{(r)}(\cdot) - x_0'(\cdot)\|_0 = 0,$$

where (i) $x_0'(\cdot) = (\overset{1}{x}_0(\cdot), u_0(\cdot), q_0'(\cdot), p_0'(\cdot))$ solves near τ, and for the sake of simplicity in $[0, \tau]$, the Cauchy problem

(14.8) $\dot{x} = f(x, \dot{u}),\ x(\tau) = (\tau, u_0(\tau), q_0(\tau), p')$ — see (f) to (g) — for $u = u_0(\cdot)$.

Since $p' \neq p_0(\tau)$, (ℓ) $q_0'(\cdot) \neq q_0(\cdot)$. In fact otherwise we had $\dot{q}_0'(\cdot) = \dot{q}_0(\cdot)$ and hence $p_0'(\cdot) = p_0(\cdot)$. By (14.7), (f), and (i), $\|q_{(r)}(\cdot) - q_0'(\cdot)\|_0 \to 0$; hence ($\ell$) implies the falsity of the relation $\|q_{(r)}(\cdot) - q_0(\cdot)\|_0 \to 0$. Then the relation $(14.2)_1$, deduced above, is also false. By this contradiction hypothesis (a) is false, i.e. the function $f(x, v)$ of x and v, is linear in v. Therefore equation $(2.2)_1$ is (at most) linear in $\dot{u}$.

By Corollary 6.1 and assertion (α) below Def. 13.3, Theor. 14.1 implies the following

Corollary 14.1 - *If* $(2.2)_1$ *with* $z = (q, p)$ *denotes the* SHE (11.6) *with* $\gamma = \tilde{\gamma}(u)$, *then its linearity in* $\dot{u}$, *the* BVC⁰-, C⁰-, *and* $\mathcal{L}^1$*-controllabilities of* u *in it, and the* BVC⁰ *and* C⁰*-semicontrollabilities of* u *in* $(2.2)_1$ *are six mutually equivalent conditions.*

Since the linearity mentioned in Corollary 14.1 holds for every (admissible) choice of $\tilde{\gamma}(\cdot)$ $(\in C^3)$ iff (11.6) is linear in $\dot{u}$, by Defs. 7.1 and Defs. 13.3-4 Corollary 14.1 implies the following

Corollary 14.2 - *The linearity of the* SHE (11.6) *in* $\dot{\gamma}$, *the* 1*-dimensional* BVC⁰-, C⁰-, *and* $\mathcal{L}^1$*-controllabilities of the (functional* M*-dimensional) parameter* γ *in* (11.6), *and its* 1*-dimensional* BVC⁰- *and* C⁰*-semicontrollabilities are six mutually equivalent conditions.*

By Theor. 13.1, the mutual equivalence of the first and sixth among the conditions considered in Corollary 14.2, implies the following.

Corollary 14.3 - *The linearity in* $\dot{\gamma}$ *of the* SHE (11.6) *is invariant under the transformations of the form* $(13.2)_{1\text{-}2}$.

This corollary has an essential application in the proof of one among the (relatively) most important theorems in [2] (Theor. 10.1) (4).

(4) The author's original (unpublished) proof of Corollary 14.3 showed directly, by means of several calculations, that the conditions (11.8) and

$$\text{(a)}\ \ \partial^3 \tilde{Q}_h / \partial\dot{\gamma}_\varrho\, \partial\dot{\gamma}_\sigma\, \partial\dot{\gamma}_\tau = 0 \quad (\varrho, \sigma, \tau = 1, \ldots, M) \text{ — see } (11.6)_3$$

are invariant under any transformation $q = q(\bar{q}, \gamma, t)$, $\gamma = \gamma(\bar{\gamma}, t)$ (slightly more general than $(13.2)_{1\text{-}2}$). Their invariance under transformations $(13.2)_{1\text{-}2}$ follows from Corollary 14.3, because by Theor. 11.1 those conditions are equivalent to the linearity of $(11.6)_{1\text{-}2}$, in $\dot{\gamma}$.

N 15 - *Semi-fitness for jumps. Its equivalence to e.g. fitness for jumps.*

The "minimal requirement" ($\mathcal{R}$) below (9.2), in Part II, which follows from the subsequent "natural requirements" (a) and (b), is considered in N 9 as compulsory in many situations, to regard the parameter u in the ODE $(9.1)_1$ as fit for jumps. When the SHE (11.6) with $\gamma = \tilde{\gamma}(u)$ is equivalent to $(2.2)_1$ — hence by (2.4), problem (2.2) is equivalent to (2.3) — and problem (2.3) has the form (9.1) — i.e. $(2.2)_1$ is a polynomial in $\dot{u}$ —, some intuitive reasons are given in N 9 for the compulsory character of ($\mathcal{R}$), through those for condition (i) in ($\mathcal{R}$) and requirement (b) below (9.3) — see ftn. 3 in N 9.

The above compulsory character might fail to be evident at first sight. In order to strengthen it or to render it quite evident, Theor. 15.2 will be proved, which by Corollary 14.2 and Theor. 15.1 below implies, among other things, the equivalence hinted at in the title. Theor. 15.2 is useful for the above purpose, in that it is based on Def. 15.1 below, which aims at introducing a requirement ($\mathcal{R}'$), which is substantially weaker than ($\mathcal{R}$) and, roughly speaking, is to ($\mathcal{R}$) (taken together with some consequences of (a) and (b) in N 9) as e.g. C^0-semicontrollability is to C^0-controllability.

Def. 15.1 - *In case the SHE* (11.6) *with* $\gamma = \tilde{\gamma}(u)$ *is espressed by* $(2.2)_1$ *and problem* (2.3), (*practically*) *equivalent to* (2.2), *has the form* (9.1), *I say that the parameter* u *in* $(2.3)_1$ *or* $(2.2)_1$ *is semifit for jumps if* (α) *for every* $\bar{x} = (\bar{t}, \bar{u}, \bar{z}) = (\bar{t}, \bar{u}, \bar{q}, \bar{p}) \in V$, *there are some* $\delta_1 (> 0)$, T_0, *and* T_1 *with* $T_0 < \bar{t} < T_1$, *such that condition* (i) *in* ($\mathcal{R}$) (*below* (9.2)) *holds, and* (β) *for every* $\epsilon > 0$ *with* $[\bar{t} - \epsilon, \bar{t} + \epsilon] \subseteq [t_0, T_1]$ *and* $\bar{B}(\bar{x}, \epsilon) \subset V$, *there is some* $\delta \in (0, \delta_1)$ *such that, first*

$$
(15.1) \qquad |j| < \delta \Rightarrow q_{Tj}(t) \in \bar{B}(\bar{q}, \epsilon) \, (\subset \mathbb{R}^N) \, \forall t \in [\bar{t}, \bar{t} + \epsilon] \, \forall T \in (\bar{t}, \epsilon),
$$

where $x(u_{Tj}, \cdot) = (\overset{1}{x}(\cdot), u_{Tj}(\cdot), q_{Tj}(\cdot), p_{Tj}(\cdot))$ *solves problem* (9.1) *in* $\Delta_\epsilon = [\bar{t} - \epsilon, \bar{t} + \epsilon]$ *for* $u = u_{Tj}(\cdot) = u_T(\cdot)$, $u_T(\cdot)$ *being the function defined by* (9.2) *in* Δ_ϵ *for* $v(t) \equiv \bar{u}\,(\bar{v} = 0)$; *and second, for some* $\eta \in (0, \epsilon)$

$$
(15.2) \qquad \lim_{T-\bar{t}-|j| \to 0^+} \| q_{Tj}(\cdot) - q_{T_0}(\cdot) \|_0^{(\eta)} = 0 \quad (T - \bar{t} > 0),
$$

$$
\| f(\cdot) \|_0^{(\eta)} = \sup \{ |f(t)| \mid \bar{t} \leqslant t \leqslant \bar{t} + \eta \}.
$$

Let us note that the difference between the present point of view and the one of Part II can be characterized by means of (15.1) in that the replacement of $q_{Tj}(t) \in \bar{B}(\bar{q}, r_1)$ in (15.1) with $(q_{Tj}(\cdot), \; p_{Tj}(\cdot)) \in \bar{B}(\bar{q}, r_1) \times \bar{B}(\bar{p}, r_1)$ renders the whole Def. 15.1 conforming with the latter point of view.

Furthermore note that $u_{TO} (= v(t) \equiv \bar{u})$ is continuous, and requirement (a) in N 9 implies the continuity of $q_{Tj}(\cdot)$'s jump w. r. t., j. By considering this continuity (not only pointwise but) in some small interval $[\bar{t}, \; \bar{t} + \eta]$, relation (15.2), condition (β) above, and hence the whole Def. 15.1 are strongly justified (without the need of additional intuitive considerations — such as those in the last two lines of ftn. 3 in N 8 — connected with the condition $p_{Tj}(t) \in \bar{B}(\bar{p}, r_1)$).

By Corollary 14.1, the BVC^0-semicontrollability of the parameter u in equation $(2.2)_1$, or $(2.3)_1$, is equivalent to its $\mathcal{L}^1$-controllability (and to the linearity of $(2.2)_1$ in $\dot u$); hence the following theorem obviously holds.

Theor. 15.1 - *The* BVC^0-*semicontrollability of the parameter* u *in the* SHE (11.6) *with* $\gamma = \tilde\gamma(u)$, *implies its semi-fitness for jumps.*

Next theorem allows, among other things, to invert the above one.

Theor. 15.2 - *Assume that* (a) *the* SHE (11.6) *with* $\gamma = \tilde\gamma(u)$ *is expressed by* $(2.2)_1$ *while problem* (2.3), *equivalent to* (2.2) *for* $x = (t, u, z) = (t, u, q, p)$, *has the* (*polynomial*) *form* (9.1), *and* (b) *the parameter* u *in it is semifit for jumps. Then equation* $(9.1)_1$ *or* $(2.2)_1$ *is linear in* $\dot u$.

Indeed fix any $\bar x = (\bar t, \bar u, \bar q, \bar p) \in V$. By (b) and Def. 15.1, for some $\delta_1 (> 0), T_0, T_1$, and $\epsilon (> 0)$, $\Delta_\epsilon = [\bar t - \epsilon, \bar t + \epsilon] \subseteq [T_0, T_1]$, (c) $\bar B(\bar x, 6\epsilon) \subset V$, and we have conditions (β) in Def. 15.1 and (i) in ($\mathcal{R}$) (below 9.2). By (β) in Def. 15.1, for some $\delta \in (0, \delta_1)$ and $\eta \in (0, \epsilon)$, (15.1-2) hold.

Step 1 - *Condition* (ii) in ($\mathcal{R}$) — *and hence* ($\mathcal{R}$) *in* N 9 — *holds.*

In fact assume that Step. 1, i.e. (ii) in N 9, is false. Then, since the falsity of (9.3) for $r = \bar r (> 0)$ implies the one for any $r \in (0, \bar r)$,

(γ) *there is some* $r \in (0, \epsilon)$ *such that* (d) $\bar B(\bar x, 5r) \subset V$, (e) $\Delta_r \subseteq [T_0, T_1]$, *and for all* $\delta \in (0, \delta_1)$ *and* $T_2 \in (\bar t, T_1) \cap (\bar t, \bar t + r)$ *we can choose some* $j \in (-\delta, \delta)$, $T \in (\bar t, T_2)$, *and* $t' \in (\bar t, T)$ *for which — remembering the definitions of* $u_{Tj}(\cdot)$ *and* $x(u_{Tj}, \cdot)$ *below* (15.1), *and that* $x(u_{Tj}, \cdot)$ *exists by condition* (i) *in* ($\mathcal{R}$) — *we have*

$$(15.3)\quad x_{Tj}(t') \notin \bar B(\bar x, 3r) \text{ with } x_{Tj}(\cdot) = x(u_{Tj}, \cdot) = (\overset{1}{x}_{Tj}(\cdot), u_{Tj}(\cdot), q_{Tj}(\cdot), p_{Tj}(\cdot)).$$

Now consider the sequence $r_h = rh^{-1}$ for $h = 4, 5, \ldots$, so that $3r_h < r$. Then, by condition (β) in Def. 15.1 and (e),

(δ) *for any* $h > 3$, *there are some* $\delta_h \in (0, \delta_1)$ *and* $\mathfrak{N}_h \in (\bar t, T_1)$, *for which*

$$(15.4)\qquad q_{Tj}(t) \in \bar B(\bar q, r_h) \quad \forall t \in [\bar t, \bar t + \epsilon] \quad \forall j \in (-\delta_h, \delta_h) \quad \forall T \in (\bar t, \mathfrak{N}_h).$$

By (γ) there are some $j_h \in (0, \delta_h) \cap (0, r_h)$, $T_h \in (\bar t, \mathfrak{N}_h) \cap (\bar t, \bar t + r_h)$, and $t'_h \in (\bar t, T_h)$ such that

$$(15.5)\qquad x_{T_h j_h}(t'_h) \notin \bar B(\bar x, 3r) \quad (0 < t'_h - \bar t < r_h, \ |j_h| < r_h, r_h = r/h).$$

Furthermore, by (15.4), $(15.4)_1$ holds for $T = T_h$, $j = j_h$, and $t = t'_h$. Hence (since $|t'_h - \bar t| + |u(t'_h) - \bar u| + |q_{Tj}(t'_h) - \bar q| < 3r_h < r$)

$$(15.6)\qquad p_{T_h j_h}(t'_h) \notin \bar B(\bar p, r) \quad (h = 4, 5, \ldots).$$

Then, by the continuity of the solution $x_{Tj}(\cdot)$ of Cauchy problem (9.1) in $\Delta_\epsilon = [\bar t - \epsilon, \bar t + \epsilon]$, for some $t_h \in (\bar t, t'_h)$

$$(15.7)\qquad p^*_h = {}_D p_{T_h j_h}(t_h) \in K = {}_D\bar B(\bar p, 2r) - \mathring{B}(\bar p, r) \quad (h > 3)$$

while, by (15.4)

(15.8) $$q_h^* = q_{T_h j_h}(t_h) \in \bar{B}(\bar{q}, r_h) \quad (h > 3).$$

Since K is compact, some subsequence $\{p_{h_r}^*\}$ of $\{p_h^*\}$ converges to some point $p' \in K$; and by (15.8) and $(15.5)_5$, $q_h^* \to \bar{q}$. Thus by $(15.5)_{2\text{-}5}$ and $(15.7)_3$

(15.9) $$\lim_{r\to\infty} (t_h, \bar{u} + j_h, q_h^*, p_h^*)_{h=h_r} = (\bar{t}, \bar{u}, \bar{q}, p') \neq (\bar{t}, \bar{u}, \bar{q}, \bar{p}).$$

By (d), i.e. $\bar{B}(\bar{x}, 5r) \subset V$, general theorems on ODEs imply the existence of some $\zeta \in [0, \eta]$ such that, for all $\tilde{t} \in [\bar{t}, \bar{t} + \zeta]$ and all $\tilde{x} = (\tilde{t}, \tilde{u}, \tilde{q}, \tilde{p}) \in \bar{B}(\bar{x}, 4r)$, setting $\tilde{v}(t) \equiv \tilde{u}$, in Δ_3 we have the solution $\bar{x}(\tilde{v}, \tilde{x}, \cdot) = (\overset{1}{x}(\cdot), \tilde{v}(\cdot), \tilde{q}(\cdot), \tilde{p}(\cdot))$ of (9.1) for $u = \tilde{v}(\cdot)$, with the initial condition $\tilde{x}(\tilde{t}) = \tilde{x}$. Let us set $v_{(r)} = v_{(r)}(t) \equiv \bar{u} + j_{h_r}$ and $x_r(\cdot) = (\overset{1}{x}(\cdot), v_{(r)}(\cdot), q_{(r)}(\cdot), p_{(r)}(\cdot)) = \bar{x}(v_{(r)}, \tilde{x}, \cdot)$ for $\tilde{x} = (t_h, v_{(r)}(t_h), q_h^*, p_h^*)$ with $h = h_r$, and $x'(\cdot) = (\overset{1}{x}(\cdot), v, q'(\cdot), p'(\cdot)) = \bar{x}(v, \tilde{x}, \cdot)$ for $\tilde{x} = (\bar{t}, \bar{u}, \bar{q}, p')$ and $v(t) \equiv \bar{u}$. Furthermore $\|v_{(r)} - v\|_1^\zeta \to 0$ $(\|f\|_1^\zeta = \|f\|_0^\zeta + \|f'\|_0^\zeta)$. Then, by $(15.9)_1$, since $|v_{(r)}(t_{h_r}) - \bar{u}| < j_{h_r}$,

(15.10) $$\|x_{(r)}(\cdot) - x'(\cdot)\|_1^\zeta \to 0, \text{ hence } \|q_{(r)}(\cdot) - q'(\cdot)\|_1^\zeta \to 0.$$

Remembering $(15.3)_{2\text{-}3}$ and that $u_{T0}(t) \equiv v(t) \equiv \bar{u}\ \forall t \in [\bar{t}, \bar{t} + \zeta]$, we see that (f) *the set* $S = \{t \in (\bar{t}, \bar{t} + \zeta) \mid q_{T0}(t') \neq q'(t)\}$ *is dense in* $[\bar{t}, \bar{t}, + \zeta]$. In fact, if $q_{T0}(\cdot)$ and $q'(\cdot)$ coincided on some open subset of $[\bar{t}, \bar{t} + \zeta]$, then the same would hold for $\dot{q}_{T0}(\cdot)$ and $\dot{q}'(\cdot)$, and hence for $p_{T0}(\cdot)$ and $p'(\cdot)$; then $x_{T0}(t) = x'(t)\ \forall t \in [0, \zeta]$, so that $\bar{p} = p_{T0}(\bar{t}) = p'(\bar{t}) = p'$, in contrast with $(15.9)_2$.

By (f), $S \neq \emptyset$ and (g) $q_{T0}(t') \neq q'(t')$ for some $t' \in [\bar{t}, \bar{t} + \zeta]$. Furthermore $q_{(r)}(t) = q_{Tj}(t)$ with $T = t_{h_r}$ and $j = j_{h_r} \forall t \in [T, \bar{t} + \zeta]$; and $t_{h_r} \in (\bar{t}, t')$ for r large. Hence, by (15.2), $|q_{(r)}(t') - q_{T0}(t')| \to 0$; and by $(15.10)_2$ $|q_{ir}(t') - q'(t')| \to 0$, which is incompatible with (g). By this absurd consequence of the falsity of condition (ii) in ($\mathcal{R}$), we can assert the truth of (ii), and hence that Step 1 holds.

Now assume that (2.3) has the form (9.1) with $\nu \geqslant 1$. Then we can deduce (9.5-6) again; and for $\nu > 1$ this is incompatible with condition (ii) above. Therefore $\nu = 1$, i.e. $(2.3)_1$, and hence $(2.2)_1$, is linear in $\dot{u}$.

q.e.d.

Theors 15.1-2 and Corollary 14.2 imply the following

Corollary 15.1 — *Assumption* (a) *in Theor.* 15.2 *implies that* (1) *the semifitness for jumps of the parameter* u *in* $(9.1)_1$, *i.e. the* SHE (11.6) *with* $\gamma = \tilde{\gamma}(u)$, (2) u*'s fitness for jumps — see Def.* 10.1—, *and* (3) *the linearity of* $(9.1)_1$ *w.r.t.* $\dot{u}$ *are three mutually equivalent conditions.*

N.16 — *On the* 1*-dimensional semifitness for jumps of the M-dimensional functional parameter* γ *in the* SHE (11.6).

Def. 16.1 — Consider the SHE (11.6). *I say that the* (*M-dimensional functional*) *parameter* γ *in it is* 1*-dimensionally semifit for jumps if for every* (*admissible*) C^3*-path* $\tilde{\gamma}(\cdot)$, *the scalar parameter* u *is semifit for jumps in equation* (11.6) *with* $\gamma = \tilde{\gamma}(u)$ (*and* $\dot{\gamma} = \tilde{\gamma}'(u)\dot{u}$).

Theor. 16.1 — *If* (α) *the Lagrangian forces* $Q_h = \tilde{\mathcal{Q}}_h(t, q, \gamma, p, \dot{\gamma})$ — *see* (11.7) — *are polynomials in* $\dot{\gamma}$, *then the* 1-*dimensional semifitness of the parameter* γ *in the* SHE (11.6), γ' s 1-*dimensional fitness for jumps there — see Def.* 10.1 — *and the linearity of* (11.6) *w.r.t.* $\dot{\gamma}$ *are three mutually equivalent conditions.*

In fact, by assumption (α), for every admissible C^3-path $\tilde{\gamma}(\cdot)$ the SHE (11.6) with $\gamma = \tilde{\gamma}(u)$ is equivalent to an instance of the ODE $(9.1)_1$, i.e. assumption (a) in Theor. 15.2 holds. Then, by Corollary 15.1, the three conditions in its consequent are mutually equivalent, for every choice of $\tilde{\gamma}(\cdot)$. Furthermore the validity of the first [last] among those conditions for every choice of $\tilde{\gamma}(\cdot)$, is equivalent to the first [last] among the three conditions, say C_1 to C_3, mentioned in the consequent of Theor. 16.1, by Def. 15.1 [by the forms $\dot{z} = \Phi(t, z, \gamma, \dot{\gamma})$ and $\dot{z} = \Phi(t, z, \gamma(u), \tilde{\gamma}'(u)\dot{u})$ taken by equation (11.6) for $z = (q, p)$ before and after the replacement $\gamma \rightarrow \tilde{\gamma}(\cdot)$ respectively]. Then Corollary 15.1 implies the mutual equivalence of conditions C_1 and C_3. The mutual equivalence of conditions C_1 and C_2 follows from Theor. 10.2.

q.e.d.

REFERENCES

[1] BRESSAN A.: *On control theory and its applications to certain problems for Lagrangian systems. On hyper-impulsive motions of these. I Some general mathematical considerations on controllizable parameters. II Some purely mathematical considerations for hyper-impulsive motions. Applications to Lagrangian systems.* Atti dell'Accad. Naz. dei Lincei

[2] BRESSAN A.: *Hyper-impulsive motions and controllizable co-ordinates for Lagrangian systems,* being printed as a memoir on Atti Accad. Lincei.