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On domains with ACC on invertible ideals

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Algebra. — *On domains with* ACC *on invertible ideals.* Nota di STEFANIA GABELLI, presentata (*) dal Socio G. ZAPPA.

ABSTRACT. - If A is a domain with the ascending chain condition on (integral) invertible ideals, then the group $\mathrm{T}(\mathrm{A})$ of its invertible ideals is generated by the set $\mathrm{I}_m(\mathrm{A})$ of maximal invertible ideals. In this note we study some properties of $\mathrm{I}_m(\mathrm{A})$ and we prove that, if $\mathrm{I}(\mathrm{A})$ is a free group on $\mathrm{I}_m(\mathrm{A})$, then A is a locally factorial Krull domain.

KEY WORDS: Krull domain; Locally factorial; Invertible ideal.

RIASSUNTO. - *Sui domini con la condizione della catena ascendente sugli ideali invertibili.* Se A è un dominio con la condizione della catena ascendente sugli ideali (interi) invertibili, allora il gruppo $\mathrm{I}(\mathrm{A})$ dei suoi ideali invertibili è generato dall'insieme $\mathrm{I}_m(\mathrm{A})$ degli ideali invertibili massimali. In questa nota si mettono in relazione alcune proprietà di $\mathrm{I}_m(\mathrm{A})$ con quelle di $\mathrm{I}(\mathrm{A})$ e si dimostra che, se il gruppo $\mathrm{I}(\mathrm{A})$ è libero su $\mathrm{I}_m(\mathrm{A})$, allora A è un dominio di Krull localmente fattoriale, ottenendo così una nuova caratterizzazione dei domini di Krull localmente fattoriali.

Throughout, A will be an integral domain. An ideal I of A is *invertible* if $\mathrm{I}(\mathrm{A}:\mathrm{I}) = \mathrm{A}$. We say that an invertible ideal is a *maximal invertible* ideal of A if it is maximal among the proper integral invertible ideals of A. We denote by $\mathrm{I}(\mathrm{A})$ the set of all invertible ideals and by $\mathrm{I}_m(\mathrm{A})$ the set of the maximal invertible ideals of A. An ideal I is *divisorial* if $\mathrm{I} = \mathrm{A}:(\mathrm{A}:\mathrm{I})$. We denote by $\mathrm{D}(\mathrm{A})$ the set of the divisorial ideals and by $\mathrm{D}_m(\mathrm{A})$ the set of *maximal divisorial* ideals of A, that is the ideals that are maximal with respect to being divisorial. An invertible ideal is divisorial.

A *Mori domain* is a domain with the ascending chain condition (in short a.c.c.) on integral divisorial ideals. If A is Mori, then by Zorn's Lemma $\mathrm{D}_m(\mathrm{A})$ and $\mathrm{I}_m(\mathrm{A})$ are not empty. Similarly if a.c.c. on integral invertible ideals holds in A, then $\mathrm{I}_m(\mathrm{A}) \neq \emptyset$. However $\mathrm{D}_m(\mathrm{A})$ and $\mathrm{I}_m(\mathrm{A})$ can be empty. For example if (A,M) is a valuation domain such that M is not principal, then $\mathrm{D}_m(\mathrm{A}) = \mathrm{I}_m(\mathrm{A}) = \emptyset$. In fact if I is any invertible ideal of A, then I is principal. Thus $\mathrm{I} \neq \mathrm{M}$ and if $x \in \mathrm{M}\setminus\mathrm{I}$, then $\mathrm{I} \subsetneq x\mathrm{A}$.

Not every domain with a.c.c. on invertible ideals is Mori.

EXAMPLE: Let $\mathrm{A} = k + X\mathrm{D} + X^2\mathrm{K}[\![X]\!]$, where $k \subset \mathrm{K}$ are two distinct fields, D is a domain, but not a field, such that $k \subset \mathrm{D} \subset \mathrm{K}$ and X is an indeterminate over

(*) Nella seduta del 14 maggio 1988.

K. A is a domain with a.c.c. on principal ideals which is not Mori [BDF, Example 17]. Since A is a quasilocal domain (with maximal ideal $M = XD + X^2K[[X]]$), then every invertible ideal of A is principal and A has a.c.c. on invertible ideals too.

A divisorial maximal ideal is prime [BG2, Proposition 1.3] and an invertible prime ideal is maximal invertible [Gi, Theorem 6.6.] and maximal divisorial, [G, Proposition 1.6]. So we easily get the following Proposition.

PROPOSITION 1: Let A be an integral domain. Then

$$\mathrm{Spec}(A)\cap I_m(A) = \mathrm{Spec}(A)\cap I(A) = D_m(A)\cap I(A).$$

A maximal invertible ideal need not be prime. For example if A is a «strongly Mori» domain [BG1, Section 2 and Example 2.6], then it is known that no maximal divisorial ideal of A is invertible. Thus, by Proposition 1, no maximal invertible ideal is prime.

The following Theorem is due to Greco [Gr, Teorema 2.1 and Corollario 2.2].

THEOREM 2: If A is a domain with a.c.c. on invertible ideals, then $I(A)$ is generated by $I_m(A)$.

Thus if A has a.c.c. on invertible ideals and I is invertible, we can write $I = H_1 \dots H_d$, where $H_i \in I_m(A)$. This factorization is not unique in general. For example, let $A = \mathbb{Q} + X\mathbb{R}[[X]]$. A is a quasilocal Mori domain, then every invertible ideal is principal and $I_m(A)$ consists exactly of the principal ideals generated by irreducible elements. In A we have $X^2 = X \cdot X = \sqrt{n}/n\, X \cdot \sqrt{n}\, X$, $n \in \mathbb{N}$. Thus the ideal X^2A has infinitely many different factorizations. In fact, if n is not a square, then $\sqrt{n} \notin A$ and $\sqrt{n}\, X$ is an irreducible element which is not associated to X. We observe that X is not prime in A. Indeed A is strongly Mori and so no irreducible element of A is prime.

PROPOSITION 3: Let A be a domain with a.c.c. on invertible ideals and let I be an integral invertible ideal of A. Then:

1) Every representation of I as a product of maximal invertible ideals is of type $I = P_1^{e_1} \dots P_n^{e_n} H_1^{r_1} \dots H_m^{r_m}$, where $P_1, \dots, P_n$ are exactly the prime invertible ideals of A containing I, $e_i \in \mathbb{N}$ is such that $I \subset P^{e_i}$ but $I \not\subset P^{e_i+1}$ for $1 \leq i \leq n$, $H_j \in I_m(A)$ is not prime and $r_j \in \mathbb{N}$ for $1 \leq j \leq m$.

2) $I = P_1^{e_1} \dots P_n^{e_n}$, where P_i is a prime invertible ideal for $1 \leq i \leq n$, if and only if I is not contained in any non prime maximal invertible ideal. In this case $I = P_1^{e_1} \dots P_n^{e_n}$ is the unique representation of I as a product of maximal invertible ideals.

Proof. 1) Let $I = H_1^{t_1} \dots H_d^{t_d}$, $H_i \in I_m(A)$, $H_l \neq H_j$ for $i \neq j$ and $t_i \in \mathbb{N}$. Suppose that P is a prime invertible ideal of A and $I \subset P$. Since H_i, P are maximal invertible ideals, then $H_i = P$ for some i and, up to order, $I = P^{t_1} H_2^{t_2} \dots H_d^{t_d}$. If $I \subset P^{t_1+1}$, then $I = P^{t_1+1} J$ [Gi, Theorem 6.2] and $PJ = H_2^{t_2} \dots H_d^{t_d} \subset P$. Contradiction.

2) Let $I = P_1^{e_1} \dots P_n^{e_n} H_1^{r_1} \dots H_m^{r_m}$ as in (1) and let H be a non-prime maximal inver-

tible ideal of A. It is clear that if $I \not\subset H$, then $H \neq H_j$ for each j. Therefore if I is not contained in any non-prime maximal invertible ideal, then $I = P_1^{e_1} \dots P_n^{e_n}$. Conversely, let $I = P_1^{e_1} \dots P_n^{e_n}$ and suppose that $I \subset H$. By [Gi, Theorem 6.2], we have that $I = HJ$ with $J \in I(A)$ and, since $H \neq P_i$ for every i, then $J \subset P_i$ for every i. It follows by (1) that $J = P_1^{s_1} \dots P_n^{s_n} K_1^{t_1} \dots K_d^{t_d}$, where $K_j \in I_m(A)$ is not prime and $s_i, t_j \in \mathbb{N}$ for every i, j. Thus $I = P_1^{e_1} \dots P_n^{e_n} = HP_1^{s_1} \dots P_n^{s_n} K_1^{t_1} \dots K_d^{t_d}$ and, again by (1), $e_i = s_i$ for $1 \leq i \leq n$. It follows that $A = HK_1^{t_1} \dots K_d^{t_d}$. Contradiction.

A *Krull domain* is a completely integrally closed Mori domain, that is a Mori domain such that $D(A)$ is a group [F, Section 3]. A domain A is said to be *locally factorial* if A_P is factorial for every prime ideal P of A. While a factorial domain is a Krull domain, a locally factorial domain need not be Krull. For example an almost Dedekind domain is locally factorial, but it is a Krull domain if and only if is a Dedekind domain [Gi, Section 29]. A locally factorial Krull domain is a Krull domain with $D(A) = I(A)$. Several equivalent conditions are known, some of them can be found for example in [A, Theorem 3.1]. A locally factorial Mori domain is Krull. In fact a locally factorial domain is completely integrally closed.

The following Theorem characterizes locally factorial Krull domains among domains with a.c.c. on invertible ideals.

THEOREM 4: Let A be a domain with a.c.c. on invertible ideals. Then the following are equivalent:

(i) $I(A)$ is a free group generated by $I_m(A)$,

(ii) $I_m(A) \subset \mathrm{Spec}(A)$,

(iii) $I_m(A) = D_m(A)$,

(iv) $I(A) = D(A)$,

(v) A is a locally factorial Krull domain.

Proof: (i) $\Rightarrow$ (ii) Suppose that $L \in I_m(A)$ and let $a, b \in A$ such that $ab \in L$ and $b \notin L$. If $aA = H_1 \dots H_r$ and $bA = K_1 \dots K_s$ with H_i, $K_j \in I_m(A)$, then $abA = H_1 \dots H_r K_1 \dots K_s$. On the other hand $abA = LJ$ with $J \in I(A)$ by [Gi, Theorem 6.2]. Since abA has a unique representation as a product of maximal invertible ideals, then, up to order, $L = H_1$ or $L = K_1$. Therefore $L = H_1$ because $b \notin L$. Hence $a \in L$ and L is prime. (ii) $\Rightarrow$ (i) by Proposition 3(2). (ii) $\Rightarrow$(v) If $I_m(A) \subset \mathrm{Spec}(A)$, then every integral invertible ideal of A is a product of prime invertible ideals by Theorem 2. Hence A is a locally factorial Krull domain by [A, Theorem 3.1]. (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) are clear. (iii) $\Rightarrow$ (ii) $D_m(A) \subset \mathrm{Spec}(A)$ by [BG1, Proposition 2.1].

REMARKS

1. We can prove directly (ii) $\Rightarrow$ (iii) in Theorem 4. In fact, if $I_m(A) \subset \mathrm{Spec}(A)$, then $I_m(A) = D_m(A) \cap I(A) \subset D_m(A)$ by Proposition 1. Let $Q \in D_m(A)$ be non-invertible and let $x \in Q, x \neq 0$. By Theorem 2, $xA = H_1 \dots H_d$, where $H_i \in I_m(A)$ for

$1 \leq i \leq d$. Since Q is not invertible, $H_i \neq Q$ for each i and, since $H_i \in D_m(A)$, then $H_i \not\subset Q$ for each i. Hence $xA \not\subset Q$, because Q is prime. Contradiction.

2. If A is a generalized GCD domain, that is the intersection of two principal ideals is an invertible ideal, then a.c.c. on invertible ideals is equivalent to A being a Krull domain and to all the conditions of Theorem 4 by [AA, Theorem 3] and [A, Theorem 3.1].

3. A locally factorial Krull domain is factorial if and only if every invertible ideal is principal. Therefore for those domains with a.c.c. on invertible ideals such that every invertible ideal is principal, the conditions of Theorem 4 are all equivalent to A being a factorial domain (cf. [F, Proposition 6.1]).

COROLLARY 5: A domain with a.c.c. on invertible ideals is a Dedekind domain if and only if $I_m(A) = \mathrm{Max}(A)$.

Proof: It is clear that if A is Dedekind, then $I_m(A) = \mathrm{Max}(A)$ [F, Theorem 13.1]. Conversely if $I_m(A) = \mathrm{Max}(A)$ then a fortiori $I_m(A) = D_m(A)$. Therefore A is Krull by Theorem 4. It follows that $I_m(A)$ is the set of all height 1 primes of A and, since $I_m(A) = \mathrm{Max}(A)$, we have that $\dim A = 1$.

Note that a domain with $I_m(A) = \mathrm{Max}(A)$ is not necessarily a Dedekind domain. For example if $A = \mathbb{Z} + X\mathbb{Q}[X]$, then $I_m(A) = \mathrm{Max}(A)$. In fact every maximal ideal of A is principal generated by a prime integer. Hovewer, A is not Dedekind because A does not have a.c.c. on principal ideals. For example the chain of ideals $\{(1/2^k)XA\}_{k \geq 1}$ is not stationary.

REFERENCES

[1] ANDERSON D.D., *Globalization of some local properties in Krull domains,* Proc. AMS, *85* (2) (1982), 141-145.

[AA] ANDERSON D.D. and ANDERSON D.F., *Generalized GCD domains,* Comment. Math. Univ. St. Pauli, *28* (2) (1979), 215-221.

[BDF] BARUCCI V., DOBBS D.E. and FONTANA M., *Conducive integral domains as pullbacks,* Manuscripta Math. *54*, (1986), 261-267.

[BG1] BARUCCI V. and GABELLI S., *How far is a Mori domain from being a Krull domain?,* J. Pure Appl. Algebra, *45* (1987), 101-112.

[BG2] BARUCCI V. and GABELLI S., *On the class Group of a Mori domain,* J. Algebra, *108* (1987), 161-173.

[F] FOSSUM R., *The divisor Class Group of a Krull domain*, Springer-Verlag, 1973.

[G] GABELLI S., *Completely integrally closed domains and t-ideals,* Boll. UMI, (7) *3*-B (1989).

[Gi] GILMER R., *Multiplicative ideal theory*, Queen's Paper in Pure and Applied Mathematics, No 12, 1968.

[Gr] GRECO S., *Sugli ideali frazionari invertibili,* Rend. Sem. Mat. Padova, *36* (1966), 315-333.