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Converging semigroups of holomorphic maps


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Abstract. – In this paper we study the semigroups $\Phi: \mathbb{R}^+ \rightarrow \text{Hol}(D, D)$ of holomorphic maps of a strictly convex domain $D \subset \mathbb{C}^n$ into itself. In particular, we characterize the semigroups converging, uniformly on compact subsets, to a holomorphic map $h: D \rightarrow \mathbb{C}^n$.

Key words: Semigroups of holomorphic maps; Convex domains; Iteration of holomorphic maps; Fixed points.

Riassunto. – Semigruppi convergenti di applicazioni olomorfe. In questa nota vengono caratterizzati quei semigruppi $\Phi: \mathbb{R}^+ \rightarrow \text{Hol}(D, D)$ di applicazioni olomorfe di un dominio strettamente convesso $D \subset \mathbb{C}^n$ in sé che convergono, uniformemente sui compatti, ad un'applicazione olomorfa $h: D \rightarrow \mathbb{C}^n$.

In 1926, Wolff and Denjoy (see [4], [8], [9] and [10]) proved the following theorem:

Theorem 0.1: (Wolff-Denjoy) Let $\Delta$ be the unit disk in the complex plane, and $f: \Delta \rightarrow \Delta$ a holomorphic function. Then the sequence $\{f^n\}$ of iterates of $f$ converges, uniformly on compact sets, to a holomorphic function $h: \Delta \rightarrow \mathbb{C}$ iff $f$ is not an automorphism of $\Delta$ with exactly one fixed point.

This very nice result can be generalized in two ways. The first one is increasing the dimension of the ambient space, that is looking at domains in $\mathbb{C}^n$, with $n \geq 1$.

For strictly convex $C^2$ bounded domains in $\mathbb{C}^n$, a complete result is known (see [1]):

Theorem 0.2: Let $D$ be a strictly convex bounded $C^2$ domain, and $f: D \rightarrow D$ a holomorphic map. Then the sequence $\{f^n\}$ of iterates converges, uniformly on compact sets, iff either

(i) $f$ has a fixed point $z_0 \in D$, and the differential $df(z_0)$ has no eigenvalues $\lambda \neq 1$ with $|\lambda| = 1$, or
(ii) $f$ has no fixed points.

It is worth noticing that, by Schwarz's lemma, if $D = \Delta$ Theorem 0.2 becomes exactly Theorem 0.1.

The second kind of generalization is changing the object of study. The main property of a sequence of iterates \( \{f^n\} \) is that \( f^m \circ f^n = f^{m+n} \) for all \( m, n \in \mathbb{N} \). So, it is very natural to investigate the properties of a semigroup of holomorphic maps in a domain \( D \subset \mathbb{C}^n \), that is of a continuous map \( \Phi : \mathbb{R}^+ \to \text{Hol}(D, D) \) such that \( \Phi_0 = \text{id}_D \), and \( \Phi_s \circ \Phi_t = \Phi_{s+t} \), for all \( s, t \in \mathbb{R}^+ \). In this paper \( \text{Hol}(D, D) \) is always endowed with the compact-open topology; by Vitali's theorem (see e.g. [6]), convergence in this topology is equivalent to punctual convergence.

Vesentini, in a series of seminars, characterized the converging semigroups in \( \Delta \) (but compare also [3]):

**Theorem 0.3:** (Vesentini) Let \( \Phi : \mathbb{R}^+ \to \text{Hol}(\Delta, \Delta) \) be a semigroup of holomorphic functions in \( \Delta \). Then \( \Phi \) converges for \( t \to \infty \) to a holomorphic function \( h : \Delta \to \mathbb{C} \) iff no \( \Phi_t \) is an automorphism of \( \Delta \) with exactly one fixed point.

The aim of this note is to extend Theorem 0.3 to strictly convex domains in \( \mathbb{C}^n \), exactly as Theorem 0.2 was a generalization of Theorem 0.1.

Let \( D \) be a strictly convex bounded \( \mathbb{C}^2 \) domain, and \( \Phi : \mathbb{R}^+ \to \text{Hol}(D, D) \) a semigroup of holomorphic maps in \( D \). We shall say that \( z_0 \in D \) is a fixed point of \( \Phi \) if \( z_0 \in \text{Fix}(\Phi_t) \) for all \( t \in \mathbb{R}^+ \), where \( \text{Fix}(\Phi_t) \) is the fixed point set of \( \Phi_t \). On the other hand, \( \Phi \) is fixed point free if \( \text{Fix}(\Phi_t) = \emptyset \) for all \( t > 0 \). An important fact we shall show later is that either \( \Phi \) has a fixed point, or \( \Phi \) is fixed point free.

The first step toward our aim is:

**Proposition 1.1:** Let \( D \) be a strictly convex bounded \( \mathbb{C}^2 \) domain, and \( \Phi : \mathbb{R}^+ \to \text{Hol}(D, D) \) a semigroup in \( D \). Assume \( \text{Fix}(\Phi_{t_0}) = \emptyset \) for some \( t_0 > 0 \). Then the semigroup converges to a constant \( x \in \partial D \).

**Proof:** In [1] it is shown that the sequence \( \{\Phi_{nt_0}\} = \{(\Phi_{t_0})^n\} \) converges, uniformly on compact sets, to a point \( x \in \partial D \).

Fix \( z_0 \in D \), and let \( K = \{\Phi_s(z_0) \mid 0 \leq s \leq t_0\} \). By continuity, \( K \) is a compact subset of \( D \); therefore, for all \( \epsilon > 0 \) there is \( n_{\epsilon} \in \mathbb{N} \) such that

\[
n \geq n_{\epsilon} \Rightarrow ||\Phi_{nt_0} - x||_K < \epsilon = \sup_{0 \leq s \leq t_0} |\Phi_{nt_0}(z_0) - x| < \epsilon,
\]

that is \( |\Phi_t(z_0) - x| < \epsilon \) for all \( t \geq nt_0 \). In other words, \( \Phi_t(z_0) \) converges to \( x \) for all \( z_0 \in D \); by Vitali's theorem (cf. [6]), \( \Phi_t \to x \), q.e.d.

**Corollary 1.2:** Let \( D \) be a strictly convex bounded \( \mathbb{C}^2 \) domain, and \( \Phi : \mathbb{R}^+ \to \text{Hol}(D, D) \) a semigroup in \( D \). Then \( \Phi \) is fixed point free iff \( \text{Fix}(\Phi_{t_0}) = \emptyset \) for some \( t_0 > 0 \).
Proof: One direction is trivial. Conversely, if \( \text{Fix}(\Phi_{t_0}) = \emptyset \) for some \( t_0 > 0 \), then by Proposition 1.1 the semigroup converges to a point in the boundary of \( D \); hence no \( \Phi_t \) with \( t > 0 \) can have a fixed point, q.e.d.

The next step is crucial:

**Proposition 1.3:** Let \( D \) be a strictly convex bounded \( C^2 \) domain, and \( \Phi: \mathbb{R}^+ \to \text{Hol}(D,D) \) a semigroup in \( D \). Assume \( \text{Fix}(\Phi_{t_0}) \neq \emptyset \) for some \( t_0 > 0 \). Then there is a non-empty closed connected submanifold \( F \) of \( D \) contained in \( \text{Fix}(\Phi_t) \) for every \( t \in \mathbb{R}^+ \). In particular, \( \Phi \) has fixed points.

**Proof:** Put \( f_n = \Phi_{t_0/2^n} \); then \( f_0 = \Phi_{t_0} \) and \( (f_{n+1})^2 = f_n \). Let \( F_n = \text{Fix}(f_n) \); by Vigué's work (cf. [7]), every \( F_n \) is a closed connected submanifold of \( D \), and \( F_n \supseteq F_{n+1} \).

Moreover, \( F_0 \neq \emptyset \); then, by Corollary 1.2 every \( F_n \) is not empty.

So we have constructed a decreasing sequence of non-empty closed connected submanifolds of \( D \); therefore \( \dim F_n \) should eventually become constant. But \( F_{n+1} \) is a closed submanifold of \( F_n \), which is connected; hence \( \dim F_{n+1} = \dim F_n \) implies \( F_{n+1} = F_n \) and the sequence \( \{F_n\} \) itself is eventually constant. Let \( F \) be its limit.

By construction, \( F \subseteq \text{Fix}(\Phi_{t_0/2^n}) \) for all \( n \in \mathbb{N} \); hence \( F \subseteq \text{Fix}(\Phi_{t_0/2^n}) \) for all \( p, n \in \mathbb{N} \). Since \( \{p/2^n, n \in \mathbb{N}\} \) is dense in \( \mathbb{R}^+ \), we finally get \( F \subseteq \text{Fix}(\Phi_t) \) for all \( t \in \mathbb{R}^+ \), q.e.d.

Corollary 1.2 and Proposition 1.3 show, as promised, that a semigroup in a strictly convex domain either has a fixed point or is fixed point free.

Proposition 1.3 is somewhat related to the following result:

**Theorem 1.4:** Let \( D \) be a convex bounded domain, and \( \mathcal{F} \subseteq \text{Hol}(D,D) \cap C^0(\overline{D}) \) a family of commuting holomorphic maps. Then \( \mathcal{F} \) has a fixed point, that is there exists \( z_0 \in \overline{D} \) such that \( f(z_0) = z_0 \) for all \( f \in \mathcal{F} \).

A proof for two maps (and \( D \) smooth) is contained in [1]; the proof of the general statement will appear in [2] and [2a].

Coming back to our problem, let \( \Phi: \mathbb{R}^+ \to \text{Hol}(D,D) \) be a semigroup with a fixed point \( z_0 \in D \). Then we can associate to \( \Phi \) the linear semigroup \( A: \mathbb{R}^+ \to \text{GL}(n, \mathbb{C}) \) given by

\[
A_t = d\Phi_t(z_0).
\]

Let \( X_\Phi \) be its infinitesimal generator; \( X_\Phi \) is called the spectral generator at \( z_0 \) of the semigroup \( \Phi \).

Since we are working in a finite dimensional space, \( A_t = \exp(tX_\Phi) \) for all \( t \in \mathbb{R}^+ \). In particular, every eigenvalue of \( A_t \) is of the form \( e^{\lambda t} \), where \( \lambda \) is an eigenvalue of \( X_\Phi \). Furthermore (see e.g. [5]), every eigenvalue of \( A_t \) is contained in \( \Delta \), and this is possible iff every eigenvalue of \( X_\Phi \) has nonpositive real part.
This is all we need for our main result:

**THEOREM 1.5:** Let $D$ be a strictly convex bounded $C^2$ domain, and $\Phi : \mathbb{R}^+ \to \text{Hol}(D, D)$ a semigroup in $D$. Then $\Phi$ converges iff either

(i) $\Phi$ has a fixed point $z_0 \in D$, and the spectral generator at $z_0$ of $\Phi$ has no nonzero purely imaginary eigenvalues, or

(ii) $\Phi$ has no fixed points.

**Proof:** If (i) holds, then for every $t_0 > 0$ the differential $d\Phi_{t_0}(z_0)$ has no eigenvalues $\lambda \neq 1$ with $|\lambda| = 1$; therefore, by Theorem 0.2, $\{\Phi_{nt_0}\}$ converges. In particular, for every fixed $p \in \mathbb{N}$ the sequence $\{\Phi_{n/2^p}\}$ converges, and the limit does not depend on $p$ – for if $p < q$ then $\{\Phi_{n/2^q}\}$ is a subsequence of $\{\Phi_{n/2^p}\}$. Since $\{n/2^p | n, p \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$, this implies that the whole semigroup converges, as we wanted to show.

If (ii) holds, then the semigroup converges by Proposition 1.1.

Conversely, assume $\Phi$ converging. Then either $\Phi$ has no fixed points, or (by Theorem 0.2) every $d\Phi_t(z_0)$ has no eigenvalues $\lambda \neq 1$ with $|\lambda| = 1$, where $z_0 \in D$ is a fixed point of $\Phi$. Hence the spectral generator at $z_0$ of $\Phi$ cannot have nonzero purely imaginary eigenvalues, and we are done, q.e.d.

Again, when $D = \Delta$ Theorem 1.5 becomes Theorem 0.3. Indeed, if $\Phi$ has a fixed point $z_0 \in \Delta$, then $\Phi_t(z_0) = e^{\lambda_t}$, where $\lambda \in \mathbb{C}$ is the spectral generator at $z_0$ of $\Phi$. By Schwarz's lemma, $\text{Re}(\lambda) \leq 0$, and $\text{Re}(\lambda) = 0$ iff every $\Phi_t$ is an automorphism of $\Delta$. Finally, $\lambda = 0$ iff $\Phi_t = \text{id}_\Delta$ for all $t \in \mathbb{R}^+$, and Theorem 0.3 is completely recovered by Theorem 1.5.

We end this note with some counterexamples showing that it is impossible to relax the hypotheses in Theorem 1.5.

Let $D = \{(z, w) \in \mathbb{C}^2 | |z|^2 + |w|^2 + |w|^2 < 3\}$. $D$ is a strictly pseudoconvex bounded smooth domain in $\mathbb{C}^2$. A semigroup in $D$ is given by

$$\Phi_t(z, w) = (z, e^{it}w).$$

$\Phi$ is fixed point free, and does not converge.

Let $D = \Delta \times \Delta$ be the bidisk in $\mathbb{C}^2$. $D$ is a convex bounded domain; a semigroup in $D$ is given by

$$\Phi_t(z, w) = \left( e^{it}z, \frac{w + \tanh(t)}{1 + \tanh(t)w} \right).$$

Again, $\Phi$ is fixed point free, and does not converge.
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REFERENCES