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On the application of control theory to certain problems for Lagrangian systems, and hyper-impulsive motion for these. I. Some general mathematical considerations on controllizable parameters

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Fisica matematica. — *On the application of control theory to certain problems for Lagrangian systems, and hyper-impulsive motion for these.* I. *Some general mathematical considerations on controllizable parameters.* Nota (*) del Corrisp. ALDO BRESSAN.

ABSTRACT. — In applying control (or feedback) theory to (mechanic) Lagrangian systems, so far forces have been generally used as values of the control $u(\cdot)$. However these values are those of a Lagrangian co-ordinate in various interesting problems with a scalar control $u = u(\cdot)$, where this control is carried out physically by adding some frictionless constraints. This pushed the author to consider a typical Lagrangian system Σ, referred to a system χ of Lagrangian co-ordinates, and to try and write some handy conditions, (C), on the coefficients of Σ's kinetic energy $\mathscr{T}$ and the Lagrangian components $\mathscr{Q}_{\mathscr{R}}$ of the forces applied to Σ, at least sufficient to satisfactorily use controls of the second kind. More specifically the conditions (C) sought for, should imply that the last M coordinates in χ are 1-*dimensionally controllizable*, in the sense that one can satisfactorily treat extremum problems concerning a class $\Gamma_{\tilde{\gamma}, \Delta, \Delta'}$ of controls $\gamma = \hat{\gamma}(t) = \tilde{\gamma}[u(t)]$ that (*i*) take as values M-tuples of values of those co-ordinates, (*ii*) have the same arbitrarily prefixed C^2-path $\tilde{\gamma}$ as trajectory, (iii) are Lebesgue integrable in that $u(\cdot) \in \mathscr{L}^1(\Delta, \Delta')$ where Δ and Δ' are suitable compact segments of $\mathbf{R}$ ($\mathring{\Delta} \neq \varnothing \neq \mathring{\Delta}'$), and (iv) are physically carried out in the above way.

One of the aims of [4] is just to write the above conditions (C) by using some recent results in control theory–see [2] where Sussmann's paper [7] is extended from continuous to measurable controls–and some consequences of them presented in [3].

The present work, divided into the Notes I to III, has in part the role of an abstract, in that the works [4] to [5] have not yet been proposed for publication and e.g. conditions (C) are written in Note II, i.e. [6], without proof. In Note I conditions (C) are shown to be necessary for the last M co-ordinates of χ to be 1-dimensionally controllizable; in doing this proof, this controllizability is regarded to include certain (relatively weak) continuity properties, which are important for checking experimentally the theory being considered, and which (therefore) are analogues of the requirement that the solutions of (physical) differential systems shoud depend on the initial data continuously. Conversely conditions (C) imply that even stronger continuity properties hold for (Σ, χ, M). The above proofs are performed in Note I from the general (purely mathematical) point of view considered in [2], by (also) using some results obtained in ([2] and) [3].

The work [4] also aims at extending the well known theory of impulsive motions, with continuous positions but with velocities suffering first order discontinuities, to a theory of hyper-impulsive motions, in which positions also suffer such discontinuities. In case the components $\mathscr{Q}_{\mathscr{R}}$ depend on Lagrangian velocities in a certain way, in Note II–see its Summary–one proves some analogues for jumps of the results on controllizability stated in Note I.

(*) Presentata nella seduta del 19 giugno 1987.

In Part III some intuitive justifications given in Part II are replaced with theorems; furthermore an invariance property is proved.

KEY WORDS: Mathematical-physics; Feedback theory.

RIASSUNTO. — *Sull'applicazione della teoria dei controlli a certi problemi per sistemi Lagrangiani e sui moti iper-impulsivi di questi.* I. *Alcune considerazioni matematiche generali sui parametri controllizzabili.* Nelle applicazioni della teoria dei controlli a sistemi (meccanici) Lagrangiani, come valori del controllo $u(\cdot)$ finora sono state usate (generalmente) delle forze. Vi sono però vari problemi interessanti in cui tali valori sono quelli di una co-ordinata Lagrangiana e il controllo è realizzato fisicamente aggiungendo vincoli lisci – v. per es. [5]. Ciò ha indotto l'autore a considerare un generico sistema Lagrangiano Σ riferito ad un generico sistema χ di coordinate Lagrangiane, e a proporsi di scrivere maneggevoli condizioni, (C), sui coefficienti dell'energia cinetica $\mathcal{T}$ di Σ e sulle componenti Lagrangiane $\mathcal{Q}_{\mathcal{R}}$ delle forze attive su esso agenti, che siano almeno sufficienti affinché controlli del secondo tipo si possano usare soddisfacentemente. Più precisamente le cercate condizioni (C) dovevano implicare la *controllizzabilità* 1-*dimensionale delle ultime* M *co-ordinate* in χ, cioè la possibilità di trattare soddisfacentemente problemi di estremo concernenti una classe $\Gamma_{\tilde{\gamma},\Delta,\Delta'}$ di controlli $\hat{\gamma}(t) \equiv \tilde{\gamma}[u(t)]$ che (i) prendono per valori M-uple di valori delle dette co-ordinate, (ii) hanno la stessa traiettoria $\tilde{\gamma}\,(\in C^2)$, fissata ad arbitrio, (iii) sono integrabili secondo Lebesgue in quanto $u(\cdot) \in \mathcal{L}^1(\Delta, \Delta')$ ove Δ e Δ' sono opportuni segmenti compatti di $\mathbf{R}\,(\mathring{\Delta} \neq \varnothing \neq \mathring{\Delta}')$, e (iv) sono realizzati fisicamente nel modo suddetto. In [4] ci si è proposti, tra l'altro, di scrivere tali condizioni (C), usando certi recenti risultati matematici in teoria dei controlli – v. [2] ove il lavoro [7] di Sussmann riguardante controlli continui, è esteso con altro metodo a controlli misurabili – e alcuni loro complementi e adattamenti esposti in [3].

Il presente lavoro, diviso nelle Note I, II e III, ha in (piccola) parte carattere preventivo, in quanto i lavori [3], [4] e [5] non sono ancora stati pubblicati e, per es., le condizioni (C) sono scritte nella Nota II senza dimostrazione. Nella Nota I si deduce che le condizioni (C) sono necessarie per la controllizzabilità 1-dimensionale delle ultime M co-ordinate in χ; e ciò si fa riguardando tale controllizzabilità come includente certe proprietà di regolarità (relativamente) deboli e analoghe al requisito che le soluzioni di sistemi differenziali (inclusi in leggi fisiche) dipendano dai dati iniziali con continuità. Viceversa le condizioni (C) implicano che anche certe più forti proprietà di continuità valgano per la terna (Σ, χ, M). Le dimostrazioni suddette si fanno nella Nota I dal punto di vista generale (puramente matematico) considerato in [2] e usando (anche) risultati ottenuti in ([2] e) [3].

Il lavoro [4] è fatto anche per estendere la ben nota teoria dei moti impulsivi, con posizioni continue ma velocità aventi discontinuità di prima specie, a una teoria dei moti iperimpulsivi, in cui (anche) le posizioni subiscono tali discontinuità. Nel caso di una certa dipendenza delle $\mathcal{Q}_{\mathcal{R}}$ dalle velocità Lagrangiane, nella Nota II, cioè [6] – v. il suo Sommario – si ottengono alcuni analoghi per le dette discontinuità, dei risultati stabiliti per la controllizzabilità nella Nota I.

Nella Parte III alcune giustificazioni intuitive date nella Parte II sono sostituite da teoremi; inoltre ivi si dimostra un teorema d'invarianza.

N. 1. Introduction (*)

As will be described in more detail in the introduction to Part 2 of this work, i.e. [6, N. 8], so far the applications of control (or feedback) theory to Lagrangian systems (generally) use only force-valued functions as controls. Instead in e.g. [5] a Lagrangian parameter (or co-ordinate) u has been rendered a control, or (briefly) *controllized*; I mean that u has been identified with $u(t)$, where the function $t \vdash u(t)$ of time t is regarded as a control. The *controllizability* of that parameter, i.e. whether or not that identification is acceptable or satisfactory, is a non-trivial problem.

The solution of e.g. a Cauchy problem including some ODEs is generally required to depend on the initial data continuously, in order to be able to check experimentally the physical correctness of the solution (and the ODEs being used). For similar reasons some analogues of that requirement must hold for Cauchy problems with a (scalar) control $u = u(t)$ — see (2.1-2) —:

$$\text{(1.1)} \qquad \dot{z} = F[t, u(t), z, \dot{u}(t)], \ z(0) = \bar{z} \ (F \in C^2(\ldots, \mathbf{R}^m), \ u(\cdot) \in C^1[0, T]).$$

A simple such analogue is the (*weak*) C^0-*controllizability* of the (functional) parameter u in $(1.1)_1$ [Def. 2.1]. In some physical situations it can be weakened, I think, into the (*weak*) BVC^0-*controllizability* of u [Def. 2.2]. The latter requirement is useful also because its validity for $(1.1)_1$, is sufficient to imply that $\dot{u}$ can occur in $(1.1)_1$ at most linearly: $F_{vv}(t, u, z, v) \equiv 0$ [Theor 3.1].

In connection with the inversion of the above result [Theorem 4.1] and the various extensions of this inversion, summarized by Corollary 6.1, let us remember that, in [7], Sussmann extends the C^1-solution $x(u, \cdot)$ of (1.1) to a weak solution, say $x(u, \cdot)$ again, with $u(\cdot) \in C^0[0, T]$ provided $(1.1)_1$ is linear in $\dot{u}$. In [2], by using different procedures, this result is (rededuced and) extended, again for $F_{vv} \equiv 0$, to e.g. the bounded measurable controls defined on the whole $[0, T]$. In case $F_{vv} \equiv 0$, the weak solution above is extended in [2] to the significant solution, $x(u, \cdot)$, for controls $u = u(\cdot)$ of the last kind—see Def. 5.1 for any F_{vv}.

The treatment of Lagrangian systems in [4] is based on [2], in part directly and in part through [3]. In [4], among other things, one determines a class of those systems, coupled with systems of Lagrangian parameters, that fulfils nice regularity requirements such as the above controllizability properties. In the present work that class is shown to be the most general one that fulfils these requirements (Theorem 3.1]. It also has stronger regularity properties.

The last assertion is proved here—also by means of a result of the work [3], in preparation—from a general point of view, by showing that, for $F_{vv} \equiv 0$,

(*) The present work has been performed within the activity sphere of the research group n. 3 of the Consiglio Nazionale delle Ricerche, in the academic year 1986-87.

the parameter u in $(1.1)_1$ is e.g. strongly C^0-controllizable [Def. 4.3, Theorem 4.2] and (strongly) $\mathscr{L}^1$-controllizable [Def. 5.2, Theorem 6.2]. The non-trivial inversion of the last result is afforded by Theorem 6.1.

Considering $\mathscr{L}^1$-controllizability is important, because it refers to possibly non-continuous (bounded measurable) controls, and these are relevant by the bang-bang character of several solutions of problems in control theory. They are also relevant for hyper-impulsive motions—see e.g. [6]—where velocities and positions can suffer discontinuities of the first kind.

The notions of 1-dimensional controllizabilities for a 1-dimensional (functional) parameter in ODEs similar to $(1.1)_1$ [Def. 7.1], and Theorem 7.1 have useful applications to Lagrangian systems.

N. 2. Controllizable (functional) scalar parameters in ODEs

Assume that

$$n = 2 + m \, , \; \mathrm{V} = \mathring{\mathrm{V}} \subseteq \mathbf{R} \times \mathbf{R} \times \mathbf{R}^m \, , \; \mathscr{V} = \mathrm{V} \times \mathbf{R} \, , \; \mathrm{F} \in \mathrm{C}^2 (\mathscr{V} \, , \mathbf{R}^m) \, . \tag{2.1}$$

Then the Cauchy problem

$$\dot{z} = \mathrm{F}(t \, , u \, , z \, , \dot{u}) \, , \qquad z(\bar{t}) = \bar{z} \tag{2.2}$$

can be reduced to the form

$$\dot{x} = f(x \, , \dot{u}) \quad , \quad x(\bar{t}) = \bar{x} \qquad (f \in \mathrm{C}^2(\mathscr{V} \, , \mathbf{R}^n)) \tag{2.3}$$

by adding the conditions and definitions

$$\dot{x}_1 = 1 \, , \; \dot{x}_2 = \dot{u}; \; x_1(\bar{t}) = \bar{t} \, , \; x_2(\bar{t}) = u(\bar{t}); \; x_{2+i} = z_i \qquad (i = 1 \, , \ldots , m) \, . \tag{2.4}$$

Relation $(2.2)_1$ can be regarded as a system of m ODEs in the $m + 1$ unknown functions z_1 to z_m and u of t, or as m ODEs in z, depending on the functional parameter $u = u(t)$ (1). After having identified u with $u(t)$—and $\dot{u}$ with $\dot{u}(t)$—in equation $(2.2)_1$, this equation appears to have $u(t)$ as a control. However is $(2.2)_1$ (regarded in this way) acceptable or satisfactory? In briefer terms can u be really rendered a control, or is u *controllizable* in $(2.2)_1$? In Def. 2.1 below some technical notions of controllizability (in this sense) are stated; and below Def. 2.2, some reasons for considering them are given, aiming in particular to show the practical importance of some among them.

(1) The parameter u in $(2.2)_1$ or $(2.3)_1$ is said to be functional, because of the presence of $\dot{u}$—see $(1.1)_1$.

DEFINITION 2.1. *The (functional scalar) parameter u in the* ODEs $(2.2)_1$ *will be said to be (weakly)* C^0*-controllizable* (2) *in case (for all* $\bar{t}, \bar{u}, \bar{z}, \bar{v}, \Delta, \Delta'$, *and* $\mathcal{U}$) (a_1) *to* (a_2) *below imply* (A) *below, i.e. in case if*

(a_1) $(\bar{x}, \bar{v}) = (\bar{t}, \bar{u}, \bar{z}, \bar{v}) \in \mathcal{V}$, Δ *and* Δ' *are compact segments of* $\mathbf{R}$, $\bar{t} \in \Delta$, $\bar{u} \in \Delta'$, *and* $\mathcal{U} \subseteq C^1(\Delta, \Delta')$, *and*

(a_2) *when* $u(\cdot) \in \mathcal{U}$, $u(\bar{t}) = \bar{u}$, *and* $\dot{u}(\bar{t}) = \bar{v}$, *the (absolutely continuous) solution* $x(u, \cdot)$ *of the Cauchy problem* (2.3) *for* $u = u(t)$—*and* $\dot{u} = \dot{u}(t)$—*is in* $C^1(\Delta, V)$, *then*

(A) $x(u, \cdot)$ *is a continuous function of* $u = u(\cdot)$ *provided the (appropriate) supnorm is used on* $\mathcal{U}$ *and* $C^1(\Delta, \mathbf{R}^n)$.

DEFINITION 2.2. *The parameter u in* $(2.2)_1$ *is said to be* BVC^0*-controllizable in case there is some* $b > 0$ *such that (for all* $\bar{t}, \bar{u}, \bar{z}, \Delta, \Delta'$, *and* $\mathcal{U}$) *condition* (A) *in* Def. 2.1 *follows from* (a_1) *to* (a_2) *in* Def. 2.1 *and the condition*

(a_3) *for some* $v_{\mathcal{U}}$, $|\dot{u} - v_{\mathcal{U}}| < b$ *for all* $u(\cdot) \in \mathcal{U}$.

Of course any C^0–controllizable (scalar) parameter is BVC^0–controllizable. Intuitively, that u in $(2.2)_1$ is C^0-[BVC^0–] controllizable means that, if for every (admissible) choice of $\bar{t}$ and $\bar{z}$ (i) we replace u in $(2.2)_1$ with a C^1–function $u_n(t)$ (and $\dot{u}$ with $\dot{u}_n(t)$) for $n \in \mathbf{N}$, (ii) the corresponding solution $x_n(\cdot)$ of the Cauchy problem (2.2) exists ($n \in \mathbf{N}$) and (iii) $\| u_n(\cdot) - u_0(\cdot) \|_0 \to 0$, where $\| \ldots \|_0$ is the sup norm [and in addition (i v) for some $\bar{v}$ independent of n (but depending on $\{u_n(\cdot)\}$) $|\dot{u} - \bar{v}| < b$], then $\| x_n(\cdot) - x_0(\cdot) \|_0 \to 0$.

Therefore (weak) C^0–controllizability is one of those regularity properties that are regarded as important to test the theory being considered, in many kinds of applications; thus, to some extent, it has in control theory, an analogue of the role played in the theory of ODEs or PDEs by the requirement that the solutions of these should depend on initial data continuously.

E.g., in case u is a Cartesian co–ordinate of a matter point P belonging to some physical system Σ, in some particular situations one may know that some a priori physical bound b holds for $|\dot{\mathrm{P}}|$ with respect to some inertial space (translating with respect to the one considered initially, with the speed $|\bar{v}|$). In these situations the BVC^0–controllizability of the parameter u suffices to assure the testability of the forecasts given by problem (2.3) regarded as involving a control $u = u(t)$.

(2) « Weak » in Def. 2.1. refers to these facts: (i) through (a_2) this definition concerns only controls in C^1, whereas Def. 4.2, of strong C^0-controllizability, refers to controls in C^0 through the condition (β_2) in it; and (ii) the existence condition (A) in Def. 4.1 has no analogue in Defs. 2.1-2.

N. 3. General form for the differential equation $(2.2)_1$ with the controllizable functional parameter u

Theorem 3.1. *If* (α) *the scalar parameter u in the* ODEs $(2..2)_1$ *is* BVCº-*controllizable, then* (β) *those equations are at most linear in $\dot{u}$.*

Proof. As an hypothesis for reduction ad absurdum, suppose that $F_{vv}(t, u, z, v) \not\equiv 0$. Then, referring to the version (2.3) of (2.2), we have $f_{vv}(\bar{x}, \bar{v}) \neq 0$ for some $(\bar{x}, \bar{v}) \in \mathscr{V}$.

For some $r_0 > 0$, the closed ball $\bar{B}((\bar{x}, \bar{v}), r_0)$ belongs to $\mathscr{V}$, and in it $f(x, v)$ has a Taylor expansion of the second order. As a consequence, for the function $\rho(x, v)$ defined in that ball by

$$f(x, v) = \alpha(x) + \beta(x)(v - \bar{v}) + 2\gamma(x)(v - \bar{v})^2 + \rho(x, v), \tag{3.1}$$

where $\alpha(x) = f(x, \bar{v})$, $\beta(x) = f_v(x, \bar{v})$, and $4\gamma(x) = f_{vv}(x, \bar{v})$, we have (3)

$$\lim_{(x, v) \to (\bar{x}, \bar{v})} \frac{\rho(x, v)}{\mathscr{N}^2} = 0, \quad \text{where } \mathscr{N}^2 = |x - \bar{x}|^2 + (v - \bar{v})^2, \mathscr{N} \geq 0. \tag{3.2}$$

For any constant $a \in (0, r_0)$ and any $\epsilon > 0$, after having put the time origin at the instant $\bar{t}\ (\bar{t} = 0)$ let us set

$$u_\varepsilon(t) = \bar{u} + \bar{v}t + \varepsilon a \sin\frac{t}{\varepsilon}, \quad \text{hence } \dot{u}_\varepsilon - \bar{v} = a \cos\frac{t}{\varepsilon}, \tag{3.3}$$

$$2(\dot{u}_\varepsilon - \bar{v})^2 = a^2\left(1 + \cos\frac{2t}{\varepsilon}\right).$$

Then $u_0(t) \equiv \bar{u} + \bar{v}t$ is the limit of $u_\varepsilon(t)$ for $\varepsilon \to 0^+$, when this limit is considered either pointwise or in the sup-norm.

Let $y_\varepsilon(\cdot)$ solve in $\mathbf{R}$ the Cauchy problem

$$\dot{x} = \alpha(\bar{x}) + \beta(\bar{x})(\dot{u}_\varepsilon - \bar{v}) + 2\gamma(\bar{x})(\dot{u}_\varepsilon - \bar{v})^2, \quad x(0) = \bar{x} \qquad (\varepsilon \geq 0). \tag{3.4}$$

(3) By using the notations A [. . .] for any linear operator A—like in [1], p. 146 – and by writing, e.g., $\bar{f}$ for $f(\bar{x}, \bar{v})$, (*a*) $f(x, v) = \bar{f} + \bar{f}_x[x - \bar{x}] + \bar{f}_v(v - \bar{v}) + 2^{-1}\bar{f}_{xx}[\bar{x} - \bar{x}, x - \bar{x}] + \bar{f}_{xv}[x - \bar{x}, v - \bar{v}] + 2^{-1}\bar{f}_{vv}(v - \bar{v})^2 + P(x, v)$ with (*b*) $\lim_{(x, v) \to (\bar{x}, \bar{v})} \mathscr{N}^2 P(x, v) = 0$. Hence $P \in C^2$ and P vanishes at $(\bar{x}, \bar{v})$ with its 1st and 2nd partial derivatives. Then, by (*a*) and (3.1), $\alpha(x) = \bar{f} + \bar{f}_x[x - \bar{x}] + 2^{-1}\bar{f}_{xx}[x - \bar{x}, x - \bar{x}] + P(x, \bar{v})$, $\beta(x) = \bar{f}_v + \bar{f}_{xv}[x - \bar{x}] + P_v(x, \bar{v})$, and $\gamma(x) = 4\bar{f}_{vv} + 4P_{vv}(x, \bar{v})$, so that $\rho(x, v) = P(x, v) - P(x, \bar{v}) - 2^{-1}P_{vv}(x, \bar{v})(v - \bar{v})^2$. This and (*b*) imply (3.2).

Hence, for $\varepsilon > 0$, $(3.3)_{2-3}$ imply that [4]

$$y_\varepsilon(t) = \bar{x} + [\alpha(\bar{x}) + a^2\gamma(\bar{x})]\,t + \varepsilon\beta(\bar{x})\,a\sin\frac{t}{\varepsilon} + \tag{3.5}$$

$$+ \varepsilon\frac{a^2}{2}\,\gamma(\bar{x})\sin\frac{2t}{\varepsilon} \quad (\varepsilon > 0)$$

and since $\dot{u}_0 = \bar{v}$, the solution of problem (3.4) for $\varepsilon = 0$ is

$$y_0(t) = \bar{x} + \alpha(\bar{x})\,t\,, \text{ hence } \lim_{\varepsilon\to 0^+} y_\varepsilon(t) = y_0(t) + a^2\gamma(\bar{x})\,t\,. \tag{3.6}$$

For $\varepsilon \geq 0$ let us now consider the maximal solution $x_\varepsilon(\cdot)$ of Cauchy problem (2.3) for $u = u_\varepsilon(t)$. Then, for t in the domain $\mathscr{D}_{x_\varepsilon(\cdot)}$ of $x_\varepsilon(\cdot)$, $\varepsilon > 0$, and $x_s \equiv x_\varepsilon(s)$, by (3.1) and $(3.3)_{2-3}$

$$\begin{aligned} x_\varepsilon(t) &= \bar{x} + \int_0^t \Big[\alpha(x_s) + \beta(x_s)\,a\cos\frac{t}{\varepsilon} + \gamma(x_s)\,a^2\left(1 + \cos\frac{2t}{\varepsilon}\right) + \\ &\qquad + \rho(x_s, \dot{u}_\varepsilon(s))\Big]\,ds\,, \\ x_0(t) &= \bar{x} + \int_0^t \{\alpha[x_0(s)] + \rho[x_0(s), \dot{u}(s)]\}\,ds \;(\dot{u}_0(s) \equiv \bar{v})\,. \end{aligned} \tag{3.7}$$

By (3.5), this yields for $t \in \mathscr{D}_{x_\varepsilon(\cdot)}$

$$\begin{aligned} |x_\varepsilon(t) - y_\varepsilon(t)| &\leq \int_0^t \{|a(x_s) - \alpha(\bar{x})| + |\beta(x_s) - \beta(\bar{x})|\,a + \\ &\qquad + |\gamma(x_s) - \gamma(\bar{x})|\,a^2 + |\rho(x_s, \dot{u}_\varepsilon(s))|\}\,ds\,, \\ |x_0(t) - y_0(t)| &\leq \int_0^t \{|\alpha[x_0(s)] - \alpha(\bar{x})| + |\rho[x_0(s), \dot{u}(s)]|\}\,ds\,. \end{aligned} \tag{3.8}$$

By assumption (α), the parameter u in $(2.3)_1$ is BVC^0-controllizable—see Def. 2.2. Hence we can consider a number $b > 0$ such that, for all $\bar{x}$, Δ, Δ', and $\mathscr{U}$, conditions (a_1) to (a_5) in Defs. 2.1-2 imply condition (A) in Def. 2.1.

(4) By $(2.4)_{1-2}$, (2.3), and (3.1), $\alpha_1(x) \equiv 1 \equiv \beta_2(x)$ and $\alpha_2(x) \equiv \beta_1(x) \equiv \gamma_r(x) \equiv 0 \equiv \rho_r(x, v)$ $(r = 1, 2)$.

Choose $\eta \in (0, |\gamma(\bar{x})|)$. Then by (3.2) there is an $r_1 \in (0, r_0) \cap (0, 1) \cap \cap (0, b)$ such that

$$\text{(3.9)} \qquad \begin{gathered} |\rho(x, v)| < 2^{-4}\xi\mathcal{N}^2 \leq \eta/8 \text{ for } \eta = \xi r_1^2, \ |x - \bar{x}| \leq r_1, \\ \text{and } |v - \bar{v}| \leq r_1 . \end{gathered}$$

Set $a = r_1 (< b)$. Then there is an $r_2 \in (0, r_1]$ such that, for $|x - \bar{x}| \leq \leq r_2$

$$\text{(3.10)} \qquad \begin{gathered} |\alpha(x) - \alpha(\bar{x})| + |\beta(x) - \beta(\bar{x})| \, a + |\gamma(x) - \gamma(\bar{x})| \, a^2 < \\ < \frac{\eta}{8}\left(< a^2 \frac{\gamma(\bar{x})}{8}\right). \end{gathered}$$

By $(3.3)_2$ $|\dot{u}_\varepsilon - \bar{v}| \leq a (= r_1)$. Hence (3.9), (3.10), and (3.8) imply that, for $t \in \mathcal{D}_{x_\varepsilon(\cdot)} \cap \mathcal{D}_{x_0(\cdot)}$ (and $\varepsilon > 0$)

$$\text{(3.11)} \qquad \begin{gathered} |x_\varepsilon(t) - y_\varepsilon(t)| < \frac{\eta t}{4}, \ |x_0(t) - y_0(t)| < \frac{\eta t}{4}, \\ \text{if } x_\varepsilon(s), \ x_0(s) \in \mathrm{B}(x, r_2) \ \forall s \in [0, t] . \end{gathered}$$

By (3.5-6) and the continuity of $x_0(\cdot)$, there is a $\tau \in (0, r_2/\eta)$ and an $\varepsilon_0 > 0$ such that $x_0(t)$, $y_\varepsilon(t) \in \mathrm{B}(\bar{x}, r_2/4)$ for $t \in (0, \tau)$ and $0 \leq \varepsilon \leq \varepsilon_0$. Hence, by (3.11), (5)

$$\text{(3.12)} \qquad \begin{gathered} |x_\varepsilon(t) - \bar{x}| < \eta\tau/4 + r_2/4 < r_2/2, \ |x_0(t) - \bar{x}| < \\ < r_2/2 \ \forall t \in [0, \tau] \cap \mathcal{D}_{x_\varepsilon(\cdot)} . \end{gathered}$$

Set $t_{(\varepsilon)} = \sup([0, \tau] \cap \mathcal{D}_{x_\varepsilon(\cdot)})$. Then, by (3.12) and (3.7), for $0 \leq \varepsilon \leq \leq \varepsilon_0$. $x_\varepsilon(t_{(\varepsilon)})$ exists and is in $\mathring{\mathrm{B}}(\bar{x}, r_2/2)$, so that the solution $x_\varepsilon(\cdot)$ would not be maximal if $t_{(\varepsilon)}$ were $< \tau$. Hence $t_{(\varepsilon)} = \tau$; moreover $[0, \tau] \subseteq \mathcal{D}_{x_\varepsilon(\cdot)}$.

From now on let $x_\varepsilon(\cdot)$ $[y_\varepsilon(\cdot)]$ denote the restriction to $[0, \tau]$ of the function denoted by $x_\varepsilon(\cdot)$ $[y_\varepsilon(\cdot)]$ up to now. Furthermore set $\mathcal{U} = \{u_\varepsilon(\cdot)\}_{0 \leq \varepsilon \leq \varepsilon_0}$.

Thus conditions (a_1) and (a_2) in Def. 2.1 hold for $\Delta = [0, \tau]$ and $\Delta' = = [-r_0, r_0]$. In addition, since $a = r_1 < b$, $(3.3)_2$ implies conditions (a_3) in Def. 2.2. Then by Def. 2.2, the BVC°–controllizability of the parameter u

(5) In fact $(3.12)_{1-3}$ hold for any t that satisfies conditions $(3.12)_4$ and $(3.11)_3$ If e.g. $(3.12)_1$ failed to hold for some $t \in [0, \tau] \cap \mathcal{D}_{x_\varepsilon(\cdot)}$, then there would be a first t with $|x_\varepsilon(t) - \bar{x}| = r_2/2$. Then $x_\varepsilon(s) \in \bar{\mathrm{B}}(x, r_2) \ \forall s \in [0, t]$. Hence (3.7) would imply the existence of $x_\varepsilon(t)$ and $(3.12)_{1-2}$ whence $r_2/2 < r_2/2$. The same absurdity can be derived in connection with $(3.12)_3$. Hence (3.12) holds.

in $(2.3)_1$ implies that

$$\lim_{\varepsilon \to 0} \sup \{ | x_\varepsilon(t) - x_0(t) | \mid t \in [0, \tau] \} = 0 . \tag{3.13}$$

However, by $(3.11)_{1-2}$ the first of the relations

$$\begin{aligned} | x_\varepsilon(t) - x_0(t) | &\geq | y_\varepsilon(t) - y_0(t) | - \frac{\eta}{2} t \geq a^2 | \gamma(\bar{x}) | t - \\ &- (| \beta(\bar{x}) | + a^2 | \gamma(\bar{x}) |) \varepsilon - \frac{\eta}{2} t \end{aligned} \tag{3.14}$$

holds; furthermore (3.5) and $(3.6)_1$ yield $(3.14)_2$. Hence *for every* $\tau_0 \in (0, \tau)$ *there is an* $\varepsilon_1 \in (0, \varepsilon_0)$ *for which—see* $(3.10)_{2-3}$

$$| x_\varepsilon(t) - x_0(t) | > (a^2 | \gamma(\bar{x}) | - \eta) t > 0 \quad \forall t \in [\tau_0, \tau_1] \; \forall \varepsilon \in (0, \varepsilon_1] . \tag{3.15}$$

This contrasts with (3.13). Hence the assumption $F_{vv}(t, u, z, v) \not\equiv 0$ is absurd, so that $\dot{u}$ occurs in $(2.2)_1$ at most linearly.

q.e.d.

N. 4. INVERSION OF THEOREM 3.1. AND EQUIVALENCE OF C^0– AND BVC^0-CONTROLLIZABILITY. STRONG C^0-CONTROLLIZABILITY

THEOREM 4.1. *If* (β) $\dot{u}$ *occurs in* $(2.2)_1$ *at most linearly, then* (α) *the parameter* u *in* $(2.2)_1$ *is* C^0–, *and hence* BVC^0*-controllizable.*

Indeed, by (β) the version (2.3) of Cauchy problem (2.2) reads

$$\dot{x} = f(x) + g(x)\dot{u} , \; \bar{x}(\bar{t}) = \bar{x} , \quad \text{where} \quad f, g \in C^2(V, \mathbf{R}^n) \; (V = \overset{\circ}{V} \subseteq \mathbf{R}^n) . \tag{4.1}$$

In order to deduce (α) let us consider the following two lemmas. The first is included in Theorem 5.1 of [3]; the second is contained in [7] or, together with Lemma 6.1, in [2], sect. 3.

LEMMA 4.1. *Assume that* K_0 *and* K_1 *are compact* (*and connected*) *subsets of* V, *and*

$$\bar{x} = (\bar{t}, \bar{u}, \bar{z}) \in \overset{\circ}{K}_1 , \; K_1 \subset \overset{\circ}{K}_0 \qquad (\text{e.g. } K_r = \overline{\overset{\circ}{K}_r} \text{ for } r = 0, 1) . \tag{4.2}$$

Then there is a compact set $K \subset \mathbf{R}^n$ *and some* $\tau > 0$, $b > 0$, K', *and* Z, *such that*

$$K_1 \subseteq K \subset \overset{\circ}{K}_0 , \; K' = \bar{u} + [-b, b] , \; [\bar{t}, \bar{t} + \tau] \times K' \times Z \subseteq K \tag{4.3}$$

$$(\text{e.g. } K = \overline{\overset{\circ}{K}})$$

and that

(H) *for every* $u(\cdot) \in C^1([\bar{t}, \bar{t}+\tau], K')$, *there is a unique solution* $x(\cdot) = x(u, \cdot) \in C^1([\bar{t}, \bar{t}+\tau], K)$ *of Cauchy problem* (4.1).

LEMMA 4.2. *Assume that* (i) $K' (\subset \mathbf{R})$ *and* $K (\subset V)$ *are (connected) compact sets,* (ii) $\bar{x} \in V$, (iii) $\mathcal{U} \subseteq C^1(\Delta, K')$ *with* $\Delta = [\bar{t}, \bar{t}+\tau]$, (iv) *for every* $u = u(\cdot) \in \mathcal{U}$, *the solution* $x(\cdot) = x(u, \cdot)$ *of problem* (4.1) *for* $u = u(\cdot)$ *exists and is in* $C^1(\Delta, K)$, *and* (v) $\overline{\mathcal{U}}_{C^0}$ *is the closure of* $\mathcal{U}$ *in* $C^0(\Delta, K)$. *Then*

(A) *for every* $u = u(\cdot) \in \overline{\mathcal{U}}_{C^0}$, *the weak (or generalized) solution* $x(\cdot) = x(u, \cdot)$ *of problem* $(4.1)_{1-2}$ *exists and is in* $C^0(\Delta, K)$ (6); *furthermore*

(B) *the weak solution* $x(\cdot)$ *depends on* $u(\cdot)$ *continuously when the sup-norm* $\|\ldots\|_0$ *is used on both* $\overline{\mathcal{U}}_{C^0}$ *and* $C^0(\Delta, K)$—*so that condition* (A) *in* Def. 2.1 *holds.*

Furthermore, to deduce (α) assume conditions (a_1) to (a_2) in Def. 2.1. By (a_1) and $(2.1)_4$, $\bar{x} \in V$: hence, for some $r_0 > 0$ and $r_1 > r_0$, (4.2) holds for $K_i = \bar{B}(\bar{x}, r_i)$ $(i = 0, 1)$. Then, by Lemma 4.1, we have the consequent of the same lemma, which includes (4.3) and condition (H). Therefore, by Lemma 4.2, conditions (A) to (B) in this hold. They imply condition (A) in Def. 2.1; and by this definition we conclude that the functional parameter u in $(4.1)_1$ is C^0–controllizable. Since $(4.1)_1$ is now $(2.3)_1$, i.e. $(2.2)_1$, thesis (α) holds.

q.e.d.

By Theorems 3.1 and 4.1 we have the following.

COROLLARY 4.1. *The linearity of the* ODEs $(2.2)_1$ *in* $\dot{u}$, *and the* C^0– *and* BVC^0*–controllizabilities of the (functional scalar) parameter* u *in them are three mutually equivalent conditions.*

The theorem above can be strengthened—see Theorem 4.2 below.

DEFINITION 4.1. *The parameter* u *in* $(2.2)_1$ *can be said to be strongly* C^0*–controllizable—see ftn.* (2)—*in case, first*

(A) *for all* $(t', u', z', v') = (x', v') \in \mathcal{V}$, $\mathbf{R}$ *contains some (compact and connected) neighbourhoods* $\mathcal{N}$ *and* $\mathcal{N}'$ *of* t' *and* u' *respectively such that*

(α) *for all* $w = w(\cdot) \in C^0(\mathcal{N}, \mathcal{N}')$, *the weak solution* $x(w, \cdot) \in C^0(\mathcal{N}, V)$ *exists—see ftn.* (6)—; *and second*

(B) *for all* $(\bar{t}, \bar{u}, \bar{z}, \bar{v}) = (\bar{x}, \bar{v})$, Δ, Δ', *and* $\mathcal{U}$, *if*

(6) That $x(\cdot) = x(u, \cdot)$ $(\in C^0(\Delta, K))$ is a weak solution of problem $(4.1)_{1-2}$ for $u = u(\cdot) \in \overline{\mathcal{U}}_{C^0}$ means that, if $u_r(\cdot) \in \mathcal{U}$ and $x_r(\cdot) = x(u_r, \cdot)$ $(r \in \mathbf{N}_*)$, then $\|x_r(\cdot) - x(\cdot)\|_0 \to 0$, $\|\ldots\|_0$ being the sup-norm.

(β_1) $(\bar{x}, \bar{v}) \in \mathscr{V}$, Δ *and* Δ' *are compact segments of* $\mathbf{R}$, $\bar{t} \in \Delta$, $\bar{u} \in \Delta'$, $\mathscr{U} \subseteq C^1(\Delta, \Delta')$, *and* $\mathscr{P} \subseteq \overline{\mathscr{U}}_{C^0}$, *and*

(β_2) *for all* $u = u(\cdot) \in \mathscr{P}$, *the weak solution* $x(u, \cdot)$ $(\in C^0(\Delta, V))$ *exists, then*

(γ) *this solution* $x(u, \cdot)$ *depends on* $u(\cdot)$ *continuously when the sup-norms are used on* $\mathscr{P}$ *and* $C^0(\Delta, V)$.

Strong C^0-controllizability obviously implies (weak) C^0-controllizability

THEOREM 4.2. *Condition* (β) *in Theorem* 4.1 *implies the strong* C^0*-controllizability of the parameter* u *in* $(2.2)_1$.

Indeed we can repeat the proof of Theorem 4.1 till the deduction of condition (H) in Lemma 4.1 (made in its last but one paragraph). Then by Lemma 4.2, conditions (A) and (B) in this hold. By Lemma 4.1, this condition (A) easily implies (A) in Def. 4.1.

In order to deduce (B) in Def. 4.1, assume (β_1) to (β_2) there. Then (γ) in Def. 4.1 is an easy consequence of (B) in Lemma 4.2. Thus, by Def. 4.1, the parameter u in $(2.2)_1$ is strongly C^0-controllizable.

q.e.d.

N. 5. $\mathscr{L}^1$-CONTROLLIZABILITY

Like in [3], on the basis of [2], for $r \in \mathbf{N}_*$ I denote the Lebesgue measure on $\mathbf{R}^r$ by λ, Dirac's measure at $a\,(\in \mathbf{R}^r)$ by δ_a, and $\lambda + \delta_{\bar{t}} + \delta_a$ by ρ. Let $\mathscr{L}^\rho_s$ [$\mathscr{L}^\rho(\Delta, A)$ where $A \subseteq \mathbf{R}^s$] be the space of the ρ-measurable mappings of Δ into $\mathbf{R}^s$ [A], endowed with the norm $\varphi \vdash \|\varphi\|_\rho$:

$$\|\varphi\|_\rho = \int |\varphi| \, d\lambda + |\varphi(\bar{t})| + |\varphi(a)| . \tag{5.1}$$

Furthermore for $\mathscr{U} \subseteq C^1(\Delta, \mathbf{R})$, where Δ is a compact segment of $\mathbf{R}$, denote the closure of $\mathscr{U}$ in $\mathscr{L}^\rho_1$ by $\overline{\mathscr{U}}_{\mathscr{L}^\rho_1}$, and set

$$\overline{\overline{\mathscr{U}}}_{\mathscr{L}^\rho_1} = \{u \mid u \in \overline{\mathscr{U}}_{\mathscr{L}^\rho_1} \ \forall a \in \Delta\} \quad (\mathscr{U} \in C^1(\Delta, \mathbf{R}), \quad \mathscr{U} \subset \overline{\mathscr{U}}_{\mathscr{L}^\rho_1} \subseteq \overline{\overline{\mathscr{U}}}_{\mathscr{L}^\rho_1} . \tag{5.2}$$

DEFINITION 5.1. *Assume that* $\Delta \subset \mathbf{R}$ *is a compact segment*, $\bar{t} \in \Delta$, *and* $u \in \overline{\overline{\mathscr{U}}}_{\mathscr{L}^\rho_1}$. *Then—following substantially* [2]*—let us say that* $x(\cdot)$ *is a* (*the*) *significant* (*generalized or weak*) *solution of Cauchy problem* (2.2) *in* Δ, *in case for all* $a \in \Delta$ *conditions* (A) *to* (B) *below hold.*(7)

(7) This notion was called " generalized solution defined pointwise " in [2], sect. 3. The notion of significant solution stated by Def. 7.1 in [3] is more complex and fit for more general situations.

(A) *If* (i) *for* $r \in \mathbf{N}_*$, $u_r(\cdot) \in C^1(\Delta, \mathbf{R})$, (ii) $\| u_r(\cdot) - u(\cdot) \|_\rho \to 0$ (i.e. $\| u_r(\cdot) - u(\cdot) \|_\lambda \to 0$ *and* $u_r(t) \to u(t)$ *for* $t = \bar{t}$ *and* $t = a$), *and* (iii) *the* (*absolutely continuous*) *solution* $x_r(\cdot) = x_r(u_r, \cdot) \in C^1(\Delta, \mathbf{R}^n)$ *of* (2.2) *in* Δ *exists* ($r \in \mathbf{N}_*$), *then* (iv) $\| x_r(\cdot) - x(\cdot) \|_\rho \to 0$.

(B) *There is some sequence* $u_1(\cdot), u_2(\cdot), \ldots$ *that satisfies conditions* (i) *to* (iv) *in* (A).

THEOREM 5.1. *Assume that* (*a*) $u(\cdot) \in \overline{\overline{\mathscr{U}}}_{\mathscr{L}_1^\rho}$, (*b*) $x(\cdot)$ *is a significant solution of the Cauchy problem* (2.2) *for* $u = u(\cdot)$, *and either* (c_1) $\tilde{x}$ *also is such a solution, or* (c_2) $u(\cdot) \in C^1(\Delta, \mathbf{R})$ *and* $\tilde{x}(\cdot) = x(u, \cdot)$, *the ordinary* (*absolutely continuous*) *solution of* (2.2) *for* $u = u(\cdot)$. *Then* $\tilde{x}(\cdot) = x(\cdot)$.

Indeed, fix $a \in \Delta$ arbitrarily and, besides (*a*) and (*b*), assume alternative (c_1). Then by (B) in Def. 5.1, (*d*) some sequence $\{u_r(\cdot)\}$ satisfies conditions (i) to (iv) in Def. 5.1, the last of which implies that (*e*) $x_r(a) \to x(a)$.

Furthermore, by (c_1) and by (A) in Def. 5.1, (*d*) implies that the analogue of (*e*) holds for $\tilde{x}(\cdot)$: (*f*) $x_r(a) \to \tilde{x}(a)$.

Now assume alternative (c_2) and set (*g*) $u_r(\cdot) = u(\cdot)$. Hence $x_r(\cdot) = \tilde{x}(\cdot)$ and (*f*) obviously holds again. In addition, (*g*) implies (i) to (iii) in Def. 5.1, so that by (A), (iv) in Def. 5.1 holds. This implies (*e*) again.

Thus, in any of the alternatives (c_1) and (c_2), (*e*) and (*f*) hold for all $a \in \Delta$; hence $\tilde{x}(\cdot) = x(\cdot)$.

q.e.d.

By this result, the significant solution of (2.2) for $u = u(\cdot) \in \overline{\overline{\mathscr{U}}}_1^\rho$, can be denoted by $x(u, \cdot)$.

DEFINITION 5.2. *The* (*functional scalar*) *parameter* u *in the* ODEs $(2.1)_1$ *will be said to be* (*strongly*) $\mathscr{L}^1$*-controllizable* (8) *in case, first,*

(A) *for all* $(t', u', z', v') = (x', v') \in \mathscr{V}$, $\mathbf{R}$ *contains some* (*compact and connected*) *neighborhoods* $\mathscr{N}$ *and* $\mathscr{N}'$ *of* t' *and* u' *respectively such that* (*for every* $w(\cdot)$)

(α) *if* $w = w(\cdot) \in \mathscr{L}^\rho(\mathscr{N}, \mathscr{N}')\ \forall a \in \mathscr{N}$, *then the corresponding significant solution* $x(w, \cdot)$ *exists and is in* $\mathscr{L}^\rho(\mathscr{N}, \mathrm{V})$; *and second*

(B) *for all* $(\bar{t}, \bar{u}, \bar{z}, \bar{v}) = (\bar{x}, \bar{v})$, Δ, Δ' *and* $\mathscr{U}$, *if —see* (2.1)

(β_1) $(\bar{x}, \bar{v}) \in \mathscr{V}$, Δ *and* Δ' *are compact segments,* $\bar{t} \in \Delta$, $\bar{u} \in \Delta'$, $\mathscr{U} \subseteq C^1(\Delta, \Delta')$, *and* $\mathscr{P} \subseteq \overline{\overline{\mathscr{U}}}_{\mathscr{L}_1^\rho}$ *—see* (5.2)*—and*

(8) "Strongly" refers to the presence in Def. 5.2 of the existence condition (A) and to the mention of the class $\mathscr{P} \subseteq \overline{\overline{\mathscr{U}}}_{\mathscr{L}_1}$ made in condition (β_2) - compare with ftn. 2.

(β_2) *for* $u(\cdot) \in \mathscr{P}$, *the significant solution* $x(\cdot) = x(u, \cdot)$ *exists and is in* $\mathscr{L}^\rho(\Delta, \mathrm{V})\ \forall a \in \Delta$, *then*

(γ) *for all* $a \in \Delta$, $x(\cdot) = x(u, \cdot)$ *depends on* $u = u(\cdot)$ *continuously when the* (*appropriate*) *norm* $\|\ldots\|_\rho$ —*see* (5.1)—*is used on* $\mathscr{P}$ *and* $\mathscr{L}^\rho(\Delta, \mathbf{R}^n)$.

N. 6. EQUIVALENCE OF $\mathscr{L}^1$-CONTROLLIZABILITY OF u WITH LINEARITY IN $\dot{u}$

THEOREM 6.1. *If* (γ) *the parameter* u *in the* ODEs $(2.2)_1$ *is* $\mathscr{L}^1$*-controllizable, then* (β) $\dot{u}$ *appears in* $(2.2)_1$ *at most linearly.*

Indeed, by assumption (γ), the part of the proof for Theorem 3.1 starting at the outset and ending with relation (3.8) (included), is still holding. At this point consider assumption (γ)—instead of (α) in Theorem 3.1. Then, since $(\bar{x}, \bar{v}) \in \mathscr{V}$, by condition (A) in Def. 5.2, for some $\mathscr{N} = \bar{t} + [-b, b]$ and $\mathscr{N}' = \bar{u} + [-b, b]$ we have condition (α) in Def. 5.2.

Now we can repeat the part of the proof for Theorem 3.1 starting with the paragraph involving (3.9) and ending with the second paragraph below (3.12). By it, from now on the domains of $x_\varepsilon(\cdot)$ and $y_\varepsilon(\cdot)$ coincide with $\Delta = [\bar{t}, \bar{t} + \tau]$ ($\varepsilon \geq 0$); Furthermore, remembering $(3.3)_1$, set

$$(6.1)\quad \mathscr{U} = \{u_{1/n}(\cdot) \mid 0 \leq n\varepsilon_0 \leq 1\,|\} \subset \mathrm{C}^1(\Delta, \mathscr{N}'), \text{ hence } \mathscr{U} \subset \overline{\overline{\mathscr{U}}}_{\mathscr{L}^\rho_1} \quad \text{—see (4.2).}$$

Since condition (α) in Def. 5.2 holds and $\Delta \subseteq \mathscr{N}$, by Theorem 5.1 the significant solution in Δ [Def. 5.1] of Cauchy problem (2.2) for $u = u_\varepsilon(\cdot)$ exists and is included in the ordinary (absolutely continuous) solution $x_\varepsilon(\cdot) = x(u_\varepsilon, \cdot)$ of (2.3) ($0 \leq \varepsilon \leq \varepsilon_0$).

Now set $\mathscr{P} = \mathscr{U}$. Then conditions ($\beta_1$) to ($\beta_2$) in Def. 5.2 hold. Then, by the assumed $\mathscr{L}^1$-controllizability of u, condition (B) in Def. 5.2 implies condition (γ) in Def. 5.2 for $\mathscr{P} = \mathscr{U}$. Hence, by $(5.1)_1$

$$(6.2)\qquad \lim_{n \to \infty} \|u_{1/n}(\cdot) - u_0(\cdot)\|_\lambda = 0 \text{—compare with (3.3).}$$

We can now rededuce (3.14) (from $(3.11)_{1-2}$, (3.5), and $(3.6)_1$) and the italicized assertion including (3.15). This assertion contrasts with (6.2). Hence the assumption $\mathrm{F}_{vv}(t, u, z, v) \equiv 0$ is absurd again. Thus our thesis (β) holds.

q.e.d.

THEOREM 6.2. *If* (β) $\dot{u}$ *occurs in* $(2.2)_1$ *linearly, then* (γ) *the parameter* u *in* $(2.2)_1$ *is* $\mathscr{L}^1$*-controllizable.*

Indeed, as well as in the proof of Theorem 4.1, by condition (β) the version (2.3) of problem (2.2) has the form (4.1). In order to deduce thesis (γ), we shall use Lemma 4.1 and the following lemma, contained in [2], sect. 3 and [3], Theor. 6.2 (*e*).

LEMMA 6.1. *Assume that* (i) $K' (\subset \mathbf{R})$ *and* $K (\subset \mathscr{V})$ *are compact sets,* (ii) $\bar{x} \in \mathscr{V}$, (iii) $\mathscr{U} \subseteq C' (\Delta , K')$ *with* $\Delta = [\bar{t} , \bar{t} + \tau]$, *and* (iv) *for every* $w = w(\cdot) \in \mathscr{U}$ *the solution* $x(w , \cdot)$ *of problem* (4.1) *exists and is in* $C^1 (\Delta , K)$. *Then condition* (B) *in Def.* 5.2 *holds and*

(C) *the significant solution* $x(\cdot) = x(w , \cdot)$ [Def. 5.1] *exists and is n* $\mathscr{L}^{\rho} (\Delta , K)$ $(\rho = \lambda + \delta_{\bar{t}} + \delta_a)$ *for all* $a \in \Delta$ *and all* $w = w(\cdot) \in \overline{\overline{\mathscr{U}}}_{\mathscr{L}^{\rho}_1}$ *—see* (5.2).

In order to deduce condition (A) in Def. 5.2, assume $(t' , u' , z' , v') = (x' , v') \in \mathscr{V} (= \mathring{\mathscr{V}})$. Then the assumptions of Lemma 4.1 obviously hold for some K_0 and K_1. Hence, for some compact set $K \subset \mathbf{R}^n$ and some $\tau > 0$, $b > 0$, K', and Z', (4.3) and condition (H) in Lemma 4.1 hold. Now Lemma 6.1 easily yields its consequent (C), which implies (α) in Def. 5.2 for $\mathscr{N} = \Delta$ and $\mathscr{N}' = K'$. By the arbitrariness of $(x , v) \in \mathscr{V}$, condition (A) in Def. 5.2 holds.

By Lemma 6.1, condition (B) in Def. 5.2. also holds; hence so does our thesis (γ).

q.e.d.

By Corollary 4.1, Theorem 4.2 and Theorems 6.1-2 yield the following

COROLLARY 6.1. *The linearity of the* ODEs $(2.2)_1$ *in* $\dot{u}$, *the* (*weak*) C^o– *and* BVC^o*–controllizabilities, and the strong* C^o– *and* $\mathscr{L}^1$*–controllizabilities of the* (*functional scalar*) *parameter u in them are five mutually equivalent conditions.*

By the equivalences above one can call *controllizability* any among the above C^o–, BVC^o–, and $\mathscr{L}^1$–controllizabilities.

N. 7. FUNCTIONAL VECTOR PARAMETERS IN ODEs, THAT ARE 1-DIMENSIONALLY CONTROLLIZABLE

In order to deal with the analogue of (2.2) for vector valued controls, consider the Cauchy problem

$$(7.1) \qquad \dot{z} = \phi (t , \gamma , z , \dot{\gamma}) , \ z(\bar{t}) = \bar{z} \ (\Gamma = \mathring{\Gamma} \subseteq \mathbf{R} \times \mathbf{R}^M \times \mathbf{R}^m \times \mathbf{R}^M , \ \phi \in C^1 (\Gamma , \mathbf{R}^m)) .$$

DEFINITION 7.1. *Let us say that the* M*–dimensional* (*functional*) *parameter* γ *in the* ODEs $(7.1)_1$ *is* 1*–dimensionally* C^o–, BVC^o–, *or* $\mathscr{L}^1$*–controllizable if, for every* C^2*–path* $\tilde{\gamma}: \mathscr{A} \to \mathbf{R}^m$ *where* $\mathscr{A} = \mathring{\mathscr{A}} \neq \phi$ *is a bounded segment* $(\subset \mathbf{R})$, *the scalar parameter u in the* ODEs $(2.2)_1$ *is* (*strongly or weakly*) C^o–, BVC^o–, *or $\mathscr{L}^1$–controllizable respectively for*

$$\text{(7.2)} \qquad \begin{gathered} F(t, u, z, v) \equiv \phi\,[t, \tilde{\gamma}(u), z, \tilde{\gamma}'(u)\, v] \\ (\textit{hence} \ \ F_{uu} = \sum_{r,s=1}^{n} \phi_{\dot{\gamma}_r \dot{\gamma}_s} \tilde{\gamma}'_r \tilde{\gamma}'_s) \,. \end{gathered}$$

By $(7.2)_2$, Def. 7.1 and Corollary 6.1 imply the following

THEOREM 7.1. *The* 1*–dimensional controllizabilities* [Def. 7.1] *of the* M*–dimensional parameter* γ *in the* ODEs $(7.1)_1$, *equivalent to the linearity in* $\dot{u}$ *of the* ODEs $(2.2)_1$ *when* $(7.2)_1$ *is in force* $\forall\, \tilde{\gamma} \in C^2(\mathscr{A}, \mathbf{R}^M)$, *hold iff equations* $(7.1)_1$ *are linear in* $\dot{\gamma}$.

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