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Equazioni a derivate parziali. — *Exact controllability of the Euler–Bernoulli equation with* $L_2(\Sigma)$*-control only in the Dirichlet Boundary condition* (*). Nota (**) di I. LASIECKA e R. TRIGGIANI, presentata dal Corrisp. R. CONTI.

ABSTRACT. — The paper studies the problem of exact controllability of the Euler–Bernoulli equation in a cylinder $\Omega \times [0, T]$ of R^{n+1}, via boundary controls acting on its lateral surface.

KEY WORDS: Exact boundary controllability; Euler–Bernoulli equation.

RIASSUNTO. — *Controllabilità esatta dell'equazione di Euler–Bernoulli con controllo frontiera in* $L_2(\Sigma)$ *agente solo nelle condizioni al contorno di Dirichlet.* Si danno condizioni per la controllabilità esatta dell'equazione di Bernoulli $w_{tt} + \Delta^2 w = 0$ in un cilindro di R^{n+1} mediante controlli sulla superficie laterale.

1. INTRODUCTION

Let Ω be an open bounded domain in R^n with sufficiently smooth boundary Γ. In Ω, we consider the following non homogeneous problem for the Euler-Bernoulli equation in the solution $w(t, x)$:

$$
(1.1) \qquad \begin{cases} (a) & w_{tt} + \Delta^2 w = 0 & \text{in } (0, T] \times \Omega \equiv Q \\ (b) & w(0, \cdot) = w^0;\ w_t(0, \cdot) = w^1 & \text{in } \Omega \\ (c) & w|_\Sigma = g_1 & \text{in } (0, T] \times \Gamma \equiv \Sigma \\ (d) & \left.\dfrac{\partial w}{\partial \nu}\right|_\Sigma = g_2 & \text{in } \Sigma \end{cases}
$$

ν unit outward normal, with control functions g_1, g_2 to be suitably selected below. In this paper, we study the problem of exact controllability for the dynamics (1.1).

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The problem of exact controllability of (1.1) with control action only in the Neumann boundary conditions

$$g_1 \equiv 0; \qquad g_2 \in L^2(\Sigma)$$

was recently studied by J.L. Lions [L. 1], where exact controllability is achieved on the space $L^2(\Omega) \times H^{-2}(\Omega)$ for $T >$ some suitable $T_0 > 0$. These results were then refined by Komornik [K. 1], who improved the estimate for T_0, and complemented by Zuazua [Z. 1], who showed that exact controllability of (1.1) on the same space is possible for arbitrarily small $T > 0$, (as expected) by adapting to present circumstances a technique, first introduced in [B - L - R. 1], to prove a needed uniqueness result. In [L. 1], J.L. Lions also raised the question as to whether problem (1.1) is exactly controllable and—if so—in what space, in the case where the control action is exercised only in the Dirichlet boundary conditions, i.e. in the case

$$g_1 \in L^2(\Sigma); \qquad g_2 \equiv 0 \tag{1.2}$$

in (1.1 *c-d*). In particular, J.L. Lions raised the question of characterizing his space F for problem (1.1) subject to (1.2). The main aim of the present note is to provide affirmative answers to these (and related) questions. Below, we shall present statements of results and we shall also provide a sketch of the proofs. For further details we refer to our forthcoming paper [L-T. 1].

2. Statement of main results

Let $A: L^2(\Omega) \supset \mathscr{D}(A) \to L^2(\Omega)$ be the (positive self-adjoint) operator defined by

$$Af = \Delta^2 f, \quad f \in \mathscr{D}(A) = H^4(\Omega) \cap H_0^2(\Omega). \tag{2.0}$$

Then, we set

$$X \equiv [\mathscr{D}(A^{1/4})]' \times [\mathscr{D}(A^{3/4})]' \tag{2.1 a}$$

$$\| x \|_X^2 \equiv \|A^{-1/4} x_1\|_\Omega^2 + \|A^{-3/4} x_2\|_\Omega^2, \quad x = [x_1, x_2] \tag{2.1 b}$$

where $\| \ \|_\Omega$ denotes the $L^2(\Omega)$-norm.

THEOREM 2.1. *Assume there exists $x_0 \in R^n$ such that*

$$(x - x_0) \cdot \nu \geq \quad \textit{constant } \gamma > 0 \quad \textit{on } \Gamma. \tag{2.2}$$

Let T; > 0 *be given arbitrary. Then : for any initial data* $(w^0, w^1) \in X$, *there exists* $g_1 \in L^2(\Sigma)$ *such that the corresponding solution of problem* (1.1), (1.2) *satisfies*

$$w(T, \cdot) = w_t(T, \cdot) = 0; \qquad \left|\begin{matrix} w \\ w_t \end{matrix}\right| \in C([0, T]; X) \qquad \square \tag{2.3}$$

Remark 2.1. By using results of Grisvard [G. 1], it can be shown that (with equivalent norms):

$$\begin{cases} a) & \mathscr{D}(A^{1/4}) = H_0^1(\Omega) \\ b) & \mathscr{D}(A^{3/4}) = \{f \in H^3(\Omega) : f|_\Gamma = \left.\dfrac{\partial f}{\partial \nu}\right|_\Gamma = 0\}. \end{cases} \tag{2.4}$$

Moreover, J.L. Lions' space F in [L. 1] can be shown in this case to coincide with $\mathscr{D}(A^{1/4}) \times \mathscr{D}(A^{3/4})$ $\square$.

THEOREM 2.2. *Under condition* (2.2) *of Theorem* 2.1, *there exists* $T_0 > 0$ *such that if* $T > T_0$ *then : for any pair* $\{w^0, w^1\} \in Y$

$$Y \equiv H_0^1(\Omega) \times H^{-1}(\Omega) \tag{2.5}$$

there exists $g_1 \in H_0^1(0, T; L^2(\Gamma))$ *such that the corresponding solution to problem* (1.1) *with such* g_1 *and* $g_2 = 0$ *satisfies* $w(T, \cdot) = w_t(T, \cdot) = 0$. $\square$

In order to relax the geometric condition (2.2) imposed on Ω, an additional control function g_2 in the Neumann boundary conditions will be used next.

THEOREM 2.3. *Given any pair of initial data* $(w^0, w^1) \in X$, *there exist boundary controls*

$$g_1 \in L^2(\Sigma) \qquad g_2 \in L^2(0, T; H^{-1}(\Gamma)) \tag{2.6}$$

such that the corresponding solution $w(t)$ *to problem* (1.1) *satisfies* $w(T, \cdot) = w_t(T, \cdot) = 0$, *where* $T > 0$ *is arbitrarily small. Moreover*

$$\left|\begin{matrix} w \\ w_t \end{matrix}\right| \in C([0, T]; X) \qquad \square$$

Remark 2.2. In the case of both Theorem 2.1 and Theorem 2.3, the space X of exact controllability coincides with the space of regularity of the solutions. In fact, applying a transposition argument to recent results of J.L. Lions [L. 2], one can show that for problem (1.1), with $w = w^1 = 0$, the map

$$(2.7) \qquad \begin{cases} [g_1, g_2] \to [w, w_t] \\ \text{is continuous: } L^2(\Sigma) \times L^2(0, T; H^{-1}(\Gamma)) \to C([0, T]; X) . \end{cases}$$

This is not the case for Theorem 2.2. □

3. Sketch of proof

3.1. Theorem 2.1.

Step 1. We use the "ontoness" approach of the operator $\mathscr{L}_T$ defined below, following the authors' work on the wave equation with Dirichlet boundary control [T. 1] and Neumann boundary control [L–T. 2]. As in these references, one can show that the solution of problem (1.1) subject to (1.2) with zero initial data is given explicitly by

$$(3.1) \qquad \left| \begin{matrix} w(t, 0; w^0 = 0, w^1 = 0) \\ \\ w_t(t, 0; w^0 = 0, w^1 = 0) \end{matrix} \right| = \mathscr{L}_T g_1 = \left| \begin{matrix} A \int_0^T S(T - t) G_1 g_1(\tau) \, d\tau \\ \\ A \int_0^T C(T - t) G_1 g_1(\tau) \, d\tau \end{matrix} \right|$$

Here G_1 is defined by

$$(3.2) \qquad G_1 g_1 = v \Longleftrightarrow \begin{cases} \Delta^2 v = 0 & \text{in } \Omega \\ v = g_1 & \text{on } \Gamma \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma \end{cases}$$

while $C(t)$ is the s.c. cosine operator generated by the negative self-adjoint operator $-A$ on $L^2(\Omega)$ and $S(t) = \int_0^t C(\tau) \, d\tau$.

Step 2. By the regularity result in Remark 2.2, (2.7), we have that $\mathscr{L}_T\colon L^2(\Sigma) \to X$. By time reversibility of problem (1.1), exact controllability of (1.1) subject to (1.2) on the space X over [0, T] means that $\mathscr{L}_T\colon L^2(\Sigma) \to$

$\rightarrow$ (onto) X; equivalently that for some $C_T > 0$, the Hilbert adjoint $\mathscr{L}_T^*$ satisfies

$$\left\| \mathscr{L}_T^* \left| \begin{matrix} z_1 \\ z_2 \end{matrix} \right| \right\|^2_{L^2(\Sigma)} \geq C_T \| \{z_1, z_2\} \|^2_X \tag{3.3}$$

Step 3. The equivalent p.d.e. version of (3.3) is the following inequality

$$\int_\Sigma \left(\frac{\partial}{\partial \nu} (\Delta \phi) \right)^2 d\Sigma \geq C'_T \| \{\phi^0, \phi^1\} \|^2_{\mathscr{D}(A^{3/4}) \times \mathscr{D}(A^{1/4})} \tag{3.4}$$

for some $C'_T > 0$ where

$$\begin{cases} (a) & \phi_{tt} + \Delta^2 \phi = 0 \\ (b) & \phi|_{t=0} = \phi^0, \phi_t|_{t=0} = \phi^1 \\ (c) & \phi|_\Sigma \equiv 0 \\ (d) & \left. \dfrac{\partial \phi}{\partial \nu} \right|_\Sigma \equiv 0 . \end{cases} \tag{3.5}$$

Step 4. The key result in the proof of Theorem 2.1 is the following Lemma which proves the Theorem's statement for sufficiently large T, at first.

LEMMA 3.1. *Under condition* (2.2), *there exists* $T_0 > 0$ *such that for all* $T > T_0$ *inequality* (3.4) *holds true with* $C'_T = c'(T - T_0)$.

Proof of Lemma 3.1. *Step* (i) We multiply (3.5 *a*) by $h \cdot \nabla(\Delta\phi)$ with $h(x) = x - x_0$, integrate by parts (Green's theorem) extensively, use the boundary conditions and obtain finally the identity:

$$\int_\Sigma \frac{\partial}{\partial \nu}(\Delta\phi)\, h \cdot \nabla(\Delta\phi)\, d\Sigma - \frac{1}{2}\int_\Sigma |\nabla(\Delta\phi)|^2 h \cdot \nu \, d\Sigma = \tag{3.6}$$

$$= \int_\Omega |\nabla\phi_t|^2 + |\nabla(\Delta\phi)|^2 dQ + \frac{n}{2}\int_\Omega \{|\nabla\phi_t|^2 - |\nabla(\Delta\phi)|^2\} dQ$$

$$- [(\phi_t, h \cdot \nabla(\Delta\phi))_{L^2(\Omega)}]_0^T .$$

with $\dim \Omega = n$.

Step (ii). We multiply (3.5 *a*) by $\Delta\phi$, again integrate by parts and use the boundary conditions. We obtain

$$\int_Q \{|\nabla\phi|^2 - |\nabla(\Delta\phi)|^2\} dQ = \left[\int_\Omega \nabla\phi \cdot \nabla\phi_t \, d\Omega \right]_0^T - \int_\Sigma \frac{\partial(\Delta\phi)}{\partial\nu} \Delta\phi \, d\Sigma . \tag{3.7}$$

Step (iii). Combining (3.6)-(3.7) one obtains

$$\text{(3.8)}\quad \int \frac{\partial(\Delta\phi)}{\partial\nu}\, h\cdot\nabla(\Delta\phi)\,d\Sigma + \frac{n}{2}\int \frac{\partial(\Delta\phi)}{\partial\nu}\,\Delta\phi\,d\Sigma - \frac{1}{2}\int |\nabla(\Delta\phi)|^2\, h\cdot \nu\,d\Sigma = \int_Q \left\{\int_\Omega |\nabla\phi_t|^2 + |\nabla(\Delta\phi)|^2\,d\Omega\right\} dt + \beta_{0,T}$$

$$\text{(3.9)}\qquad \beta_{0T} = \frac{n}{2}\left[\int_\Omega \nabla\phi\cdot\nabla\phi_t\,d\Omega\right]_0^T - [(\phi_t, h\cdot\nabla(\Delta\phi))_\Omega]_0^T .$$

Step (iv). Multiplying (3.5) by $A^{1/2}\phi_t$ and integrating over Q yields

$$\text{(3.10)}\qquad \|A^{1/4}\phi_t(t)\|_\Omega^2 + \|A^{3/4}\phi(t)\|_\Omega^2 = \|\{\phi(t),\phi_t(t)\}\|_Z^2 \equiv \|\{\phi^0,\phi^1\}\|_Z^2$$

for all $t\in R$,

$$\text{(3.11)}\qquad Z = \mathscr{D}(A^{3/4})\times\mathscr{D}(A^{1/4})$$

corresponding to the standard fact that the operator

$$\begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix},$$

which describes the dynamics of (3.5), generates a s.c. *unitary* group (on $\mathscr{D}(A^{1/2})\times L^2(\Omega)$, hence) on Z. The key observation now is that for $f = [f_1, f_2]\in Z$ (in particular for the solution ϕ of (3.5)) we have:

the norm

$$\text{(3.12)}\qquad \|f\|_Z^2 = \|A^{3/4}f_1\|_\Omega^2 + A^{1/4}f_2\|_\Omega^2$$

is *equivalent* to the norm

$$\text{(3.13)}\qquad \int_\Omega \nabla(\Delta f_1)|^2 + |\nabla f_2|^2\,d\Omega .$$

While the norm (3.12) with $f_1=\phi$, $f_2=\phi_t$ is time invariant, see (3.10), the same is not true for the norm in (3.13).

Step (v). Invoking assumption (2.2) and (3.12)-(3.13) one obtains

$$\text{(3.14)}\qquad C_{1,\varepsilon}\int_\Sigma \left(\frac{\partial(\Delta\phi)}{\partial\nu}\right)^2 d\Sigma + n\varepsilon\, C_2\int_0^T \|A^{3/4}\phi\|_\Omega^2\,dt \geq \text{left hand side of (3,8)}$$

for ε sufficiently small so that $(C_h\varepsilon - \gamma/2) < 0$, with $2\,C_h = \max|h|$ over $\bar\Gamma$.

Step (vi). By Poincaré inequality, equivalence (3.12)-(3.13) and identity (3.10) we have

$$|\beta_{0,T}| \leq C_{h,n} \| \{\phi^0, \phi^1\} \|_Z^2 \tag{3.15}$$

for the term in (3.9). This bound along with the equivalence (3.12)–(3.13) and the identity (3.10) then yields

Right hand side of (3.8)

$$\geq C_3 \int_0^T \| \{\phi, \phi_t\} \|_Z^2 \, dt - C_{h,n} \| \{\phi^0, \phi^1\} \|_Z^2 = \tag{3.16}$$

$$= C_3 T \| \{\phi^0, \phi^1\} \|_Z^2 - C_{h,n} \| \{\phi^0, \phi^1\} \|_Z^2 .$$

Combining (3.14) and (3.16) then gives

$$C_{1,\varepsilon} \int_\Sigma \left(\frac{\partial (\Delta \phi)}{\partial \nu} \right)^2 d\Sigma \geq (C_3 - n\varepsilon c) T \| \{\phi^0, \phi^1\} \|_Z^2 - \tag{3.17}$$

$$- c_{h,n} \| \{\phi^0, \phi^1\} \|_Z^2$$

from which Lemma 3·1 follows, by taking $\varepsilon > 0$ small □

Step (vii). Lemma 3.1 proves Theorem 2.1 for T sufficiently large. To obtain T arbitrarily small, one then uses this preliminary result in an argument, whose idea was introduced in [B - L - R. 1]. It consists in showing that the space of solutions of (3.5) which in addition satisfy the condition $\left.\dfrac{\partial (\Delta \phi)}{\partial \nu}\right|_\Sigma \equiv 0$ is finite dimensional. □

3.2. THEOREM 2.2. *In this case we consider the operator* $\mathscr{L}_T$ *given by* (3.1) *from* $H_0^1(0, T; L^2(\Gamma))$ *onto* $Y = H_0^1(\Omega) \times H^{-1}(\Omega) = \mathscr{D}(A^{1/4}) \times [\mathscr{D}(A^{1/4})]'$; *equivalently*

$$\left\| \mathscr{L}_T^* \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \right\|^2_{H_0^1(0, T; L^2(\Gamma))} \geq C_T \| \{z_1, z_2\} \|_Y^2 \tag{3.18}$$

counterpart of (3.3). *The p.d.e. version of* (3.18) *is now*

$$\int_\Sigma \frac{\partial}{\partial \nu} (\Delta \phi) + \frac{1}{T} \frac{\partial}{\partial \nu} \Delta [(C(T) - I) A^{-1} \phi^1 + S(T) \phi^0])^2 \, d\Sigma \geq \tag{3.19}$$

$$\geq C_T' \| \{\phi^0, \phi^1\} \|_Z^2$$

where ϕ *solves problem* (3.5) *with*

$$\phi^0 = A^{-1/2} z_1 \in \mathscr{D}(A^{3/4}); \; \phi^1 = A^{-1/2} z_2 \in \mathscr{D}(A^{1/4}) . \tag{3.20}$$

Since

$$(3.21) \quad \frac{1}{T^2} \left\| \frac{\partial}{\partial \nu} \Delta [(C(T) - I) A^{-1} \phi^1 + S(T) \phi^0] \right\|^2_{L^2(\Sigma)} \leq \frac{C}{T} \left\| \{\phi^0, \phi^1\} \right\|^2_Z$$

by taking $T > T_0$, *for sufficiently large* $T_0 > 0$, *then* (3.19) *follows from Lemma* 3.1. □

3.3. THEOREM 2.3. *By techniques similar to those in sections* 3.1 *and* 3.2 *we can show that the key inequality to establish is now*

$$(3.22) \quad \int_\Sigma \left(\frac{\partial (\Delta \phi)}{\partial \nu} \right)^2 d\Sigma + \int_\Sigma |\nabla (\Delta \phi)|^2 d\Sigma + \int_\Sigma |\Delta \phi|^2 d\Sigma \geq C_T \| \{\phi^0, \phi^1\} \|^2_Z$$

for ϕ *solution of* (3.5), *with* Z *as in* (3.11) *and with* $c_T > 0$. *That* (3.22) *holds can be shown by following the pattern of the proof of Lemma* 3.1. *The presence of the term* $|\nabla (\Delta \phi)|$ *on the left hand side of* (3.22) *allows one to dispense with geometrical conditions on* Ω *(except for smoothness of* Γ*). Moreover, use of a compacntess argument combined with classical Holmgren Uniqueness Theorem yields that* T *can be taken arbitrarily small* □

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