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Meccanica. — *On the nonlinear theory of beams with open thin sections.* Nota (*) del Socio PLACIDO CICALA (**).

ABSTRACT. — Analysis of beam with thin open sections as cylindrical shells evidences restrictions of the Wagner–Vlasof theory: these mainly concern the fulfillment of end conditions.

For the case of large deflections, the resultant equations from asymptotic analysis are presented. Their application to buckling under pure flexure shows various novel aspects. By a simple direct approach, investigation is pursued beyond the critical state: the buckled configuration turns out to be stable even for laxer constraints than usual in constructions.

KEY WORDS: Thin-walled beams; Flexural buckling; Post-critical stability.

RIASSUNTO. — *Sulla teoria non lineare delle travi con sezione aperta sottile.* Lo studio delle travi con sezione aperta sottile nell'ambito della teoria lineare dei gusci mostra le intrinseche limitazioni della teoria di Wagner–Vlasof, in particolare nell'attuazione delle condizioni al contorno. L'estensione della trattazione a grandi spostamenti è limitata alla presentazione delle equazioni risultanti. La loro applicazione all'esame degli stati critici sotto pura flessione indica diversi aspetti non segnalati nella formulazione in uso. Inoltre, spingendo l'indagine oltre lo stato critico con un semplice metodo diretto, si è constatata la stabilità di quello stato pur in condizioni di vincolo meno stringenti che le ordinarie nelle costruzioni.

Technical advantages of beam design with thin open sections are well known. Linear analysis of these structural elements was started by Timoshenko for double T sections, then pursued by Wagner and completed by Vlasof for arbitrary profile of sections. A brief account will be given here on results of asymptotics [1] furnishing a straightforward justification of the hypotheses on which pioneer scientists based linear analysis. The behaviour of these beams under large deflections has been the object of recent investigations. Results of asymptotic nonlinear analysis [2] are presented in the following, with complements to both previous papers.

1. LINEAR ASYMPTOTIC ANALYSIS OF CYLINDRICAL BEAM-SHELLS

Straight thin section beams, as well as barrel vaults, are cylindrical shells. Their geometry is defined by their midsurface where the axial coordinate ξ and the circumferential s are established as only independent variables. The open

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section, a simply connected domain in planes $\xi = \text{const.}$, concentrates along a curve λ (profile) which should be regular, with a definite curvature $\rho(s)$: irregularities (angular points, bifurcations, truncations) create local perturbations in the stress field whose three-dimensional analysis is usually left out. In asymptotic formulation the vanishing parameter is introduced as $\delta = h/b$ where h, b are longitudes measuring wall thickness and general dimensions of profile. The graphic representation of fig. 24, p. 81 [1], shows that solutions with smooth distribution along the profile contain, besides boundary layers, two classes of "significant approximation" in Eckhaus' terminology [3]: edge effects with length of ξ variation $L_{-1} \approx b\delta^{0,5} = (bh)^{0,5}$ and a "degenerated edge effects", as called by Goldenveizer, with length $L_1 \approx b\delta^{-0,5} = b^{1,5} h^{-0,5}$. Each of these classes is governed by a fourth order system in ξ: combination of their homogeneous solutions allows us to comply with four conditions for each λ point on end sections, provided their distance is $\gg L_1$. The load terms, including constraint effects, dictate "forced" lengths of ξ variation L_f. For any $L_f > > L_1$ the resultant equation governing the non-homogeneous solutions to be associated with the above homogeneous ones contains the same fundamental terms represented by the segment DP of fig. 24: therefore those formulations do not substantially differ from each other. For $L_f \approx L_2 = b\delta^{-1} = b^2/h$ the classification by order of magnitude of the unknowns is effected in fig. 26. The essential property hence emerging is the predominance of rigid motion in the displacements on the section plane. The main stresses are related to the section warping according to Wagner-Vlasof: these axial stresses have the same order as the St. Venant tangential stresses. On the contrary, for the class corresponding to L_1, examined on fig. 25, the largest stresses are related to axial elongations and to changes in the curvature ρ of λ. Calculation of these solutions leads to an eigenvalue problem: along the line λ an 8.th order differential system holds, to be associated with 4 homogeneous conditions on end points (longitudinal shell edges).

These results show that the validity field of the invariance property of the profile is restricted to beams of length $\gg L_1$: even for these it may be invalid in the neighbourhood of terminal sections and load discontinuities. Furthermore, even if the end sections are reinforced by stiff ribs ensuring local invariance of the profile form, when along the beam a distribution of transverse loading presents a length of ξ variation $\leq L_1$ the assumptions of the Wagner-Vlasof theory are violated. This furnishes an incomplete "interior" solution (as membrane theory): the point-wise position of boundary conditions is essentially dictated by it, except for some discrete parameters.

2. The bases for nonlinear asymptotic analysis of beams

Let $\vec{i}_1, \vec{i}_2, \vec{i}_3$ be a triad of orthogonal unit vectors. To establish a reference for the deformed configuration D of the beam, introduce a curve Λ whose position vector is written $\vec{\Lambda}(\xi) = \vec{i}_1 x_1 + \vec{i}_2 x_2 + \vec{i}_3 x_3$ with ξ as coordinate along Λ: differentiations $d/d\xi$ are denoted by apex. Thus we put $\vec{\Lambda}' = \vec{t}(1 + \varepsilon_0)$ where $\vec{t}$ is the unit vector tangent to Λ. Along Λ a triad of unit vector $\vec{j}_1, \vec{j}_2, \vec{j}_3 = \vec{t}$ is introduced and the rotation gradients $\tilde{\omega}_1 = \vec{j}_2' \cdot \vec{j}_3$, $\tilde{\omega}_2 = \vec{j}_3' \cdot \vec{j}_1$, $\tilde{\omega}_3 = \vec{j}_1' \cdot \vec{j}_2$ are calculated. For instance, this may be the natural triad of Λ with $\vec{j}_2$ as main normal: in this case $\bar{\omega}_1$ is the main curvature while $\tilde{\omega}_2 = 0$ and $\tilde{\omega}_3$ is the natural twist. For any choice, the vectors of the local triad $\vec{j}$ depend on ξ through its functions $\vec{x}_1, \vec{x}_2, \vec{x}_3$. The position vector of the body point in the deformed configuration D is written in the form

$$\vec{P} = \vec{\Lambda} + \vec{e}_1\xi_1 + \vec{e}_2\xi_2 + \vec{u} \tag{1}$$

with $\vec{e}_1 = \vec{j}_1 \cos\theta + \vec{j}_2 \sin\theta$, $\vec{e}_2 = \vec{j}_2 \cos\theta - \vec{j}_1 \sin\theta$. The function $\vec{u}$ of the material coordinates $\xi_1, \xi_2, \xi_3 = \xi$ represents the relative displacement with respect to the " skeleton " constituted by the curve Λ and the *beam* triad $\vec{e}_1$, $\vec{e}_2, \vec{t}$.

In a reference configuration 0, where the beam centroidal axis is given by $\vec{\Lambda}_0 = \vec{i}_3\xi$, the point (1) is represented by $\vec{\Lambda}_0 = \vec{i}_1\xi_1 + \vec{i}_2\xi_2$. In this state, the sections, loci of material points $\xi = \text{const.}$, are plane and parallel, with principal inertia axes $\xi_1 = 0$, $\xi_2 = 0$.

The definition (1) of the beam kinematics is superabundant insofar as, for every section, 4 variables (x_1, x_2, x_3, θ) are introduced in addition to $\vec{u}$. Therefore, 4 conditions can be imposed on u for any section (e.g., $\int \vec{u}\, d\xi_1\, d\xi_2 = 0$ plus an angular condition). The form (1) leads to distinguish unknowns of different orders of magnitude: correspondingly, it separates a one-dimensional problem (the calculation of the 4 additional variables) from the three dimensional problem of determining $\vec{u}$. Because of the relative smallness of this unknown, the second problem is linear even in the presence of large skeleton displacements and reduces to two dimensions in the first approximation step.

One-dimensional variables are introduced also for beam statics by writing the stress resultant $\vec{N}$ and the resultant moment $\vec{M}$ of stresses on the section with respect to the Λ point in the form

$$\vec{N} = \vec{e}_1 N_1 + \vec{e}_2 N_2 + \vec{e}_3 N_3 \quad , \quad \vec{M} = \vec{e}_1 M_1 + \vec{e}_2 M_2 + \vec{e}_3 M_3 \tag{2}$$

The equilibrium conditions for the beam element $d\xi$, subjected to the external force $\vec{p}\,d\xi$ and to the moment $\vec{q}\,d\xi$ with respect to the Λ point write

$$\vec{N}' + \vec{p} = 0 \quad , \quad \vec{M}' + \vec{\Lambda}' \times \vec{N} + \vec{q} = 0 \,. \tag{3}$$

Though these equations may be readily integrated by writing conditions of global equilibrium on beam portions at finite ξ intervals, use of Eqs. (3) is often convenient. Substituting expressions (2) in (3) leads to 6 equations for the 6 unknowns (shear forces N_1, N_2, axial force N_3, bending moment M_1, M_2, torsional moment M_3). The coefficients of the unknowns are given by

$$(4) \qquad \omega_1 = \vec{e}_2' \cdot \vec{e}_3 = \tilde{\omega}_1 \cos\theta + \tilde{\omega}_2 \sin\theta \quad , \quad \omega_2 = \tilde{\omega}_2 \cos\theta - \tilde{\omega}_1 \sin\theta \quad ,$$

$$\omega_3 = \tilde{\omega}_3 + \theta' .$$

The connection between the discrete static and kinematic variables is established through elasticity consideration of the continuum. The element of sides $\vec{i}_\alpha \, d\xi_\alpha$ $(\alpha = 1, 2, 3)$ in state 0 is represented by the element $(\overrightarrow{\partial P / \partial \xi_\alpha}) \, d\xi_\alpha$ in state D. Here the hypothesis of "stiff" material intervenes: the form change of the element is so slight that, locally, the Cartesian stress and strain components may be entered in the elasticity relationship

$$(5) \qquad \Delta\sigma_{\alpha\beta} = E_{\alpha\beta\gamma\delta} \, \Delta\varepsilon_{\gamma\delta}$$

while the large motion of the element is taken into account by deriving the strain-displacement relationships by use of the covariant tensor $(\overrightarrow{\partial P / \partial \xi_\gamma}) \cdot (\overrightarrow{\partial P / \partial \xi_\delta})$ and by locating the stresses $\sigma_{\alpha\beta}$ as contravariant components on the element in state D. The symbols Δ in (5) indicate that, if transition from state 0 to a given state U creates strains ε^* and stresses σ^*, Eq. (5) holds with the notation $\Delta f = f - f^*$ for strain and stress components. In particular, if state U is unstressed we may write $\sigma_{\alpha\beta} = E_{\alpha\beta\gamma\delta} (\varepsilon_{\gamma\delta} - \varepsilon^*_{\gamma\delta})$. Thus the first approximation solutions are obtained when the transitions $0 \to U$, $0 \to D$ involve elongations $\leq \varepsilon$. If this is not the case, a cartesian reference may be established in state U for $\sigma(\varepsilon)$ relations [2].

3. First approximation nonlinear theory of thin open section beam

In [1] the relative orders of magnitude have been determined for the infinite unknowns of the linear two-dimensional problem: in [2] (fig. 6) the absolute orders of magnitude were determined for a restricted number of unknowns, according to the needs for nonlinear analysis of thin open section beams. The results are in agreement as the same class of states is considered, with longitudinal dimensions $L = L_2 = b^2/h$. As vanishing parameter we take here the maximal admissible strain ε, with reference to stiff materal: rubberlike behaviour is ruled out. The beam is designed to sustain the Euler critical load within the limit strain: therefore $b^2/L^2 \approx \varepsilon$ and hence $\delta = h/b = b/L = \sqrt{\varepsilon}$. The maximal shear strain $h\theta/L$ takes the order ε for finite angles θ. The same order is taken by the flexural strains bx_1/L^2: therefore the deflections x_1, x_2 must be $\approx L\sqrt{\varepsilon}$: the same condition holds for the deflections x_1^*, x_2^* in the unstressed state if the same formulation is to hold for beams with small initial curvature: in this state U, the pretwist may be finite. Thus the approximate expressions

$$\vec{j}_1 = \vec{i}_1 - \vec{i}_3 x_1' \quad , \quad \vec{j}_2 = \vec{i}_2 - \vec{i}_3 x_2' \quad , \quad \vec{t} = \vec{i}_1 x_1' + \vec{i}_2 x_2' + \vec{i}_3 \tag{6}$$

may be adopted for the local triad: hence we get $\tilde{\omega}_1 = -x_2''$, $\tilde{\omega}_2 = x_1''$, $\tilde{\omega}_3 = = x_1'' x_2'$. This twist is negligible: therefore $\omega_3 = \theta'$.

The asymptotic analysis developed on the above basis leads to the relations

$$N_3/E = \Delta\,(A\varepsilon_0 + I_3\omega_3^2/2) \tag{7}$$

$$M_1/E = \Delta\,(I_1\omega_1 - H_2\omega_3' + I_{32}\omega_3^2/2)\,,\; M_2/E = \Delta\,(I_2\omega_2 + H_1\omega_3' - I_{31}\omega_3^2/2) \tag{8}$$

with

$$1 + \varepsilon_0 = x_3' + x_1'^2/2 + x_2'/2\,.$$

With respect to the expressions (4.21) of [2] here we introduce the consideration of initial deformations, with the sign Δ to signify that the deformation factors enter as differences of D and U values. The section characteristics are

$$\begin{gathered} A = \int h\,ds \quad , \quad I_1 = \int \xi_2^2 h\,ds \quad , \quad I_2 = \int \xi_1^2 h\,ds \quad , \quad I_3 = \int r^2 h\,ds \\ I_{31} = \int r^2 \xi_1 h\,ds \quad , \quad I_{32} = \int r^2 \xi_2 h\,ds \quad H_1 = \int \xi_1 \Omega h\,ds \quad , \quad H_2 = \int \xi_2 \Omega h\,ds \end{gathered} \tag{9}$$

Here ξ_1, ξ_2 represent the coordinates measured along the midline λ and $r^2 = \xi_1^2 + \xi_2^2$. Integrations extend to the whole line; Ω is the usual sector area defined by $d\Omega = \xi_1\,d\xi_2 - \xi_2\,d\xi_1$, $\int \Omega h\,ds = 0$.

The twisting moment is constituted by three summands: *a*) the moment of St. Venant tangential stresses, expressed by $GJ\Delta\omega_3$: within the present approximation we may write $3\,J = \int h^3\,ds \cdot b$) The moment due to the flow of tangential stresses along λ which can be calculated from the integral over $\lambda : \int \Omega\partial\,(h\sigma_{33})/\partial\xi\,ds$ where σ_{33} is the mean value within h of the stress in the direction $\partial\vec{P}/\partial\xi$, given by $E\Delta\,(\varepsilon_0 - \xi_1\omega_2 + \xi_2\omega_1 - \Omega\omega_3' + r^2\omega_3^2/2)$. *c*) The moment due to obliquity of these stresses with respect to the direction $\vec{e}_3$ expressed by $\int r^2\omega_3\sigma_{33}h\,ds$. Hence we find the espression [1]

$$(10)\qquad M_3 = \Delta\,(GJ\omega_3 + EH_2\omega_1' - EH_1\omega_2' - EC\omega_3'') + \omega_3\Delta\,(EI_3\varepsilon_0 - EI_{31}\omega_2 + EI_{32}\omega_1 + EI_{33}\omega_3^2/2) + \omega_3^{*\prime}\Delta\omega_3 E\int r^2\Omega h\,ds$$

with

$$C = \int \Omega^2 h\,ds \quad , \quad I_{33} = \int r^4\,h\,ds\,.$$

In the expressions (7), (8), (10) the nonlinear terms derive from the elongation terms $x_1'^2/2$, $x_2'^2/2$ in ε_0 and from the helicoidal deformation of the longitudinal fibres: this creates elongations $r^2\Delta\omega_3^2/2$ and slopes $r\omega_3$ of the deformed fibres with respect to $\vec{e}_3$. The above expressions are in agreement with Algostino's paper [4] where the case when λ is a straight line (hence $\Omega = 0$) has been dealt with. Some studies in the literature contain the mentioned additional terms, together with less significative summands: in fact, as those investigations do not follow the asymptotic approach, they may not attain the maximal simplifications compatible with the prescribed accuracy. In a recent paper [5] the expressions for stress resultants contain various terms in addition to those of the above. They stem from a spurious term, the last summand in Eq. (15) of that paper: this corresponds to the contribution $\rho r^2\theta^2\theta'$ (in the present notation) included in the shear strain $(\partial\vec{P}/\partial\xi)\cdot(\partial\vec{P}/\partial s)$. For instance, in the case of uniform helicoidal deformation of the beam, this term containing the factor θ^2 (2-2 cos θ before approximation) violates the property of helicoidal symmetry that the strain must possess.

In a more recent paper [6] the twisting moment is given an expression that, after correcting certain obvious printing errors, agrees with Eq. (10) for the initially straight beam with section having the shear centre coinciding with the centroid, as for the Z section referred to in the paper.

From the order of magnitude relations, some simplifications appear in the equilibrium equations. Hence and from (3), for $\vec{p} = \vec{q} = 0$, eliminating N_1, N_2 yields

$$(11)\qquad \begin{aligned} M_1'' - 2\,\omega_3 M_2' - \omega_3' M_2 - \omega_3^2 M_1 &= \omega_1 N_3\ , \\ M_2'' + 2\,\omega_3 M_1' + \omega_3' M_1 - \omega_3^2 M_2 &= \omega_2 N_3 \end{aligned}$$

$$(12)\qquad M_3' + \omega_1 M_2 - \omega_2 M_1 = 0 \quad , \quad N_3 = \text{const.}$$

(1) Reference to centre of shear simplifies the linear terms, not the complete form.

Substituting the expressions for stress resultants and expressing ε_0, ω_1, ω_2, ω_3 in terms of x_1, x_2, x_3, θ leads to the set of 4 equations for these 4 kinematic unknowns.

4. ELASTIC STABILITY OF THE BEAM WITH THIN OPEN SECTION UNDER PURE BENDING

The buckling problem for these beams under axial thrust was completely solved by Kappus 50 years ago. For bending, the problem still deserves some considerations.

Assume that the structure is symmetric with respect to the plane $\vec{i}_2$, $\vec{i}_3$: then $H_2 = I_{31} = 0$: this plane contains the beam axis ($x_1^* = 0$) without pretwist ($\theta^* = 0$) in state U. In the same plane a couple M is applied to the end point $x_1 = x_2 = 0$, $x_3 = l$ and contrasted at the end point $x_1 = x_2 = x_3 = 0$. The only deformation is represented by $\Delta x_2'' = -M/EI$: also $x_2^{*\prime\prime}$ will be assumed to be constant. To examine an equilibrium state in the neighbourhood of this D configuration, ω_2 and ω_3 will be handled as small quantities. Thus Eqs. (11) give $M_1'' = 0$, $M_2'' + M_1\theta'' = 0$: hence, if $M_2 = \theta = 0$ at both end points, it follows

$$M_1 = M \quad , \quad M_2 = -M\theta . \tag{13}$$

Eqs. (8) take the form

$$M_1 = EI_1(\omega_1 - \omega_1^*) \quad , \quad M_2 = EI_2\omega_2 + EH_1\theta'' . \tag{14}$$

The linearized form of Eq. (12) when substituting from (10), (13), (14) is

$$\begin{aligned} GJ\theta'' + H_1(M\theta/I_2 + EH_1\theta''/I_2)'' - EC\theta'''' + \\ + MI_{32}\theta''/I_1 = M\theta(M/EI_1 - M/EI_2 + \omega_1^*) - MH_1\theta''/I_2 . \end{aligned} \tag{15}$$

If the end conditions allow the solution $\theta = \sin(\pi\xi/l)$, Eq. (15) yields

$$\begin{aligned} \pi^2 GJ/l^2 + \pi^4 EC_m/l^4 + \pi^2 M(I_{32}/I_1 + 2H_1/I_2)/l^2 = \\ = M^2(1/EI_2 - 1/EI_1) - M\omega_1^* \end{aligned} \tag{16}$$

where

$$C_m = C - H_1^2/I_2 .$$

The two roots for M furnished by Eq. (16) give the critical moments for which the buckling can start provided the end constraints nullify the magnitudes M_2, θ, θ''.

In order to ascertain the stability of the buckled configuration we must examine the effect that a small deflection exerts on the equilibrium value of M. We limit this analysis to the section with double symmetry ($H_1 = I_{32} = 0$); furthermore we put $\omega_1^* = 0$. Reverting to Eq. $(11)_1$ we substitute the first approximation values (13) in the second order terms: thus we get $M_1'' = -M\theta'^2 - M\theta\theta''$ and hence

$$M_1 = M(1 - \theta^2/2) \tag{17}$$

Substituting this value of M_1 and the value $(13)_2$ for M_2 leads from $(11)_2$ to the equation $M_2'' = M\theta\theta'^2 - M\theta''(1 - \theta^2/2)$ and hence

$$M_2 = -M\theta(1 - \theta^2/6)\,. \tag{18}$$

Now write Eq. (12) taking account of the simplified relationships

$$M_1 = EI_1\omega_1\,, \quad M_2 = EI_2\omega_2\,, \quad M_3 = GJ\omega_3 - EC\omega_3'' + EI_{33}\omega_3^3/2$$

Thus Eq. (15) takes the modified form (3)

$$GJ\theta'' - EC\theta'''' + 3\,EI_{33}\theta'^2\,\theta''/2 = M^2(1/EI_1 - 1/EI_2)\,\theta\,(1 - 2\,\theta^2/3) \tag{19}$$

Let $\theta_0 = \sin(\pi\xi/l)$: the solution of (19) is written $\theta = \alpha\theta_0 + \chi$ where α is a vanishing parameter and $\chi = 0(\alpha)$ the relatively small correction. Eq. (19) may be put in the form $L(\theta) + C(\theta^3) = 0$ separating cubic from linear terms: for these we have that $L(\theta_0) = K\theta_0$ with $K = 0$ as characteristic equation, analogous to (16). Since both θ and θ_0 comply with the boundary conditions $\theta = \theta'' = 0$ we get $\int \theta_0 L(\theta)\, d\xi = \int \theta L(\theta_0)\, d\xi = K \int \theta\theta_0\, d\xi$. Consequently, from the identity $\int \theta_0 (L + C)\, d\xi = 0$ follows the modified form of (16)

$$K + \int \theta_0 C(\theta^3)\, d\xi \Big/ \int \theta\theta_0\, d\xi = 0\,.$$

(2) This equation agrees with (5.11) of [2], except for the N_1 term, omitted in those relationships because, erroneously, the assumption $H_2 = 0$ dictated by symmetry has been replaced by $H_1 = 0$.

(3) The exact solution of (11) for $N_3 = 0$ is $M_1 = M\cos\theta$, $M_2 = -M\sin\theta$. For a longer beam (e.g., $L \simeq b^3/h^2$) the first summand in (19) predominates in the left member: then (19) reduces to $\theta'' = a\sin 2\theta$. The first integral is $\theta'^2 = c - a\cos 2\theta$: the resolution is furnished by elliptic functions as in the Euler problem. The analogy ensures a long range of stability, because the two omitted summands have stabilizing effects.

In a first approximation evaluation of the correction, one may replace θ by $\alpha\theta_0$. Thus, by elementary computation we find

$$\pi^2 GJ/l^2 + \pi^4 EC/l^4 + 3\, EI_{33}\pi^4\alpha^2/8\, l^4 = M^2\,(1/EI_2 - 1/EI_1)\,(1 - \alpha^2/2) \tag{20}$$

Both α terms indicate that, while the deflection increases the couple needed to establish equilibrium grows: this shows that the configuration of initial post-critical phase is stable.

5. CONCLUSIONS

The asymptotic linear analysis of thin section beams as cylindrical shells shows that the Wagner-Vlasof theory may be derived from three-dimensional elasticity as a system governing interior solutions with forced length of ξ variation b^2/h.

Point-wise position of boundary conditions leads to complex calculations, introducing, among edge effects, homogeneous solutions modifying the form of the profile. In fact, these shell problems were tackled in earlier literature [7] by cumbersome iteration methods correcting the assumption of rigid sections. Nonetheless, for engineering applications to beams, the Wagner-Vlasof is helpful, allowing handy computations, provided deviations from the basic assumptions are avoided as far as possible (e.g., by stiff ribs at constraint sections): need for its extension to non-linearity has been long felt. The first approximation form, derived [2] by asymptotic approach is here modified, including effects of large pretwist. Refinements should be left to discretized computations taking account of non-linearity and profile flexure.

Application of the obtained formulation to buckling under pure flexure leads to elucidate certain aspects of this classical question. Essentially, the structure of the formula for the critical moment is $(\pi/l)\sqrt{GJEI_2}$, as found by Prandtl in his thesis. Strictly, this applies to a thin doubly symmetric section with a straight profile, whose minimal bending stiffness and torsional stiffness have both the order bh^3 (4): if the thickness/width ratio is assigned the value $\sqrt{\varepsilon}$, as above, buckling occurs when the maximal strain attains a value $\approx \varepsilon\sqrt{\varepsilon}$, showing a poor utilization of the material strength. A better design is achieved by adopting a profile that raises the minimal bending stiffness to the order b^3h: then the stiffness due to warping adds to GJ the term $\pi^2 EC_m/l^2$. Derivation and applications of this formulation is given in most textbooks following Timoshenko's and Vlasov's treatises [8], [9]. Though the ratio of the principal moments of inertia is now increased, the reduction justified by the smallness of I_2/I_1 is generally maintained. The above analysis leads to

(4) The present nonlinear theory holds for $I_1 \simeq I_2$. Applying it to the case $I_2 = o\,(I_1)$ is equivocal as then $L\omega_2$ may be finite and (6) are invalid.

find a magnifiying factor $\sqrt{I_1/(I_1 - I_2)}$ (5) in the critical moment, showing that this buckling form may be ruled out if the stiffness ratio is not small. A recent monograph [10] overlooks this aspect of the problem because of an unrealistic assumption: the beam axis is assumed to be straightened in the buckling situation. This simplification excludes from investigation also the effects of initial curvature of the beam. The completed analysis for sections having the only symmetry axis in the plane of the couple leads to Eq. (16): hence a considerable reduction in the beam strength is found for flexure contrasting the curvature ω_1^* if I_{32}, H_1, ω_1^* have the same signs.

A simple calculation permits the examination of the stability of the buckling configuration: this example shows that the straigthforward equilibrium approach may be more expeditious than the traditional energy methods. Stability of the initial buckled deflection is readily proved (6).

Some specifications concerning the constraint situation taken in consideration are in order. The conditions referring to M_2 and θ may be actuated by two end cylindrical hinges with axes along $\vec{e}_2$. The condition $\theta'' = 0$ nullifying the end "bimoments" is often adopted in the literature: the presence of the normal stresses due to helicoidal deformation of the longitudinal fibers makes justification of this statement rather difficult. However, as this constraint system appears to confer large freedom to beam ends, its adoption may be deemed as a choice tending to lower the critical value, exaggerating on the safe side. Note that the application mode of the main action M does not influence the critical value: on the contrary, if the loading does work in connection with the buckling deflections θ or x_1 the critical value would be modified. On the other side, when examining the post-critical stability, the conclusions may change according as the bending couple is applied with a constant value or its value depends on the end rotations [given by $x_2'(0) = \int M_1 \, d\xi / 2 \, EI_1$. Therefore the scheme of the constraints adopted should be specified. Each end section is connected through a Cardan joint to a rigid shaft with axis $\vec{i}_3$: one of these is fixed, the other may shift along its axis: this freedom ensures the condition $N_3 = 0$. The cylindrical hinges directly connected to the shafts have constant direction $\vec{i}_1$: the other hinges take the directions of the local axes $\vec{e}_2$. This mechanism should be completed by two ideal hinges nullifying the end bimoments. The couples exerted between the beam ends and the adjoining shafts may take prescribed values and keep them constant. When M attains

(5) Some Authors [11], [12] determine the critical bending for large curvature beams with thin rectangular sections. The above mentioned magnifying factor $\sqrt{I_1/(I_1 - I_2)}$ emerges from this analysis. For comments on this problem, see [2], p. 73 and p. 89.

(6) Stability at large may be proved by analogy with the Euler problem (S. footnote to Eq. (19). These conclusions hold for deformations that do not modify the profile: local buckling should be examined, say, by a discretization envolving profile flexure.

the value of a root of (16), an infinitesimal deflection starts in the form above calculated but cannot grow until the couple is increased: the described scheme presents post-buckling stability.

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