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Projective invariant metrics and open convex regular cones. I

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Geometria differenziale. — *Projective invariant metrics and open convex regular cones.* I. Nota di FABIO PODESTÀ, presentata (*) dal Corrisp. E. VESENTINI.

ABSTRACT. — In this work we give a characterization of the projective invariant pseudometric P, introduced by H. Wu, for a particular class of real C^∞-manifolds; in view of this result, we study the group of projective transformations for the same class of manifolds and we determine the integrated pseudodistance p of P in open convex regular cones of $\mathbf{R}^n$, endowed with the characteristic metric.

KEY WORDS: Projective connections; Regular cones; Projective transformations.

RIASSUNTO. — *Metriche invarianti proiettive e coni aperti convessi regolari.* In questo lavoro, suddiviso in una Nota I ed in una nota II, si fornisce una caratterizzazione della pseudometrica proiettiva P, introdotta da H. Wu, per varietà con connessione lineare il cui tensore di Ricci è parallelo e semidefinito negativo. Come applicazione si studiano le trasformazioni proiettive di tali varietà e la pseudodistanza p, associata a P, nei coni aperti, convessi, omogenei di $\mathbf{R}^n$. Si stabilisce infine un teorema di struttura per il gruppo delle trasformazioni affini dei coni.

§ 0. INTRODUZIONE

The purpose of this work is to find a characterization of the projective differential pseudometric P, introduced by H. Wu ([12]), on a class $\mathscr{G}$ of C^∞-manifolds endowed with a symmetric complete connection with parallel and negative semidefinite Ricci tensor.

The work is divided in two parts: in part I we show (Theorem 2.1) that for manifolds belongings to the class $\mathscr{G}$ the pseudometric P can be essentially expressed in terms of the Ricci tensor. Thanks to this result a new proof can be given of a well known theorem (Theorem 2.2) concerning projective transformations of those manifolds. These results are then applied to the case of open, convex, regular and selfadjoint cones in R^n, endowed with the characteristic Riemannian metric: under further hypotheses of irriducibility and affine-homogeneity, we have described the projective pseudodistance p, introduced by S. Kobayashi ([5]), through the study of a foliation $\{F_\lambda\}_{\lambda\in\mathbf{R}^*_+}$ of the cone, which completely determines its geometry (Theorem 3.4).

(*) Nella seduta del 13 dicembre 1986.

In part II, which will be published soon, we identify a class of projective automorphisms of the cones as a subgroup of the full group of projective transformations (Theorem 4.1), showing how in this case the word "projective" used for this theory is deeply related to its classical meaning.

In the end we prove a structure Theorem (Theorem 5.3) regarding the group of affine transformations of a selfadjoint, affine-homogeneous and irreducible cone and conclude this section establishing an analogue of the Schwarz Lemma for affine transformations (Theorem 5.4).

§ 1. Preliminaries

For all the results exposed in this section we refer to Eisenhart ([2]) and to Bortolotti ([1)]. Throughout the following, M will be a differentiable (i.e. C^∞) manifold of dimension $n \geq 2$. Two symmetric connections Γ and Γ^* on the tangent bundle to M are said to be projectively equivalent if they define the same system of geodesics up to parametrization.

If (Γ^i_{jk}) and (Γ^{*i}_{jk}) are the local components of Γ and Γ^*, the two connections are projectively equivalent if and only if there exists a global differentiable 1-form ϕ expressed locally by $\phi = \sum_j \phi_j \, dx^j$ such that

$$\forall i, j, k = 1 \dots n \qquad \Gamma^i_{jk} = \Gamma^{*i}_{jk} + \delta^i_j \phi_k + \delta^i_k \phi_j \, . \tag{1.1}$$

If we denote with ∇ the covariant differentiation relative to Γ and with R R* the respective Ricci tensors, we have locally

$$\forall i, j = 1 \dots n \qquad R^*_{ij} = R_{ij} - n \phi_{ij} + \phi_{ji} \, , \tag{1.2}$$

where

$$\phi_{ij} = \nabla_j \phi_i - \phi_i \phi_j \, . \tag{1.3}$$

Let $I = \{u \in \mathbf{R} \mid -1 < u < 1\}$ and consider on the tangent bundle TM the connection Γ.

Definition 1.1. *A differentiable map $f : I \to M$ with nowhere vanishing derivative is said to be a projective map if f is a geodesic in* M *and u is a projective parameter for this geodesic.*

We recall that if t is an affine parameter for a geodesic γ a projective parameter p is defined as a solution of the differential equation

$$\{p, t\} = \frac{2}{n-1} \sum_{ij} R_{ij} (\gamma(t)) \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \tag{1.4}$$

where $\{\,,\}$ is the Schwarz derivative and (R_{ij}) are the local components of the Ricci tensor. In ([12]) Wu has defined as follows an infinitesimal projective pseudometric P somewhat similar to the infinitesimal pseudometric on complex manifolds, introduced by Royden (see e.g. Franzoni-Vesentini ([3])): consider on I the hyperbolic Poincaré metric given by

$$(1.5) \qquad \forall u \in I \qquad ds^2 = \frac{du^2}{(1-u^2)^2}$$

and for $v \in \mathbf{R}$, $b \in I$ let $|v|_b$ the norm of v with rispect to ds^2 at b.

For $x \in M$, $X \in TM_X$, the length $P(x, X)$ of X at x with respect to Wu's projective pseudometric is

$$(1.6) \qquad P(x, X) = \inf \left\{ |V|_b \;\middle|\; \begin{array}{l} \text{there exists } f: I \to M \text{ pro . such} \\ \text{that } f(b) = x \text{ and } df_b(V) = X\,. \end{array} \right\}$$

Since (I, ds^2) is homogeneous under the action of all Moebius transformations of I onto itself and the projective parameters are invariant under Moebius transformations, we can restate (1.6) as follows

$$(1.7) \qquad P(x, X) = \inf \left\{ |V|_0 \;\middle|\; \begin{array}{l} \text{there exists } f: I \to M \text{ proj. such} \\ \text{that } f(0) = x \text{ and } df_0(V) = X \end{array} \right\}.$$

The map $P: TM \to [0\ \infty)$ is upper-semicontinuous (Wu ([12])).

S. Kobayashi ([5]) has introduced a projective pseudodistance p as follows: let ω be the distance induced on I by ds^2 and pick $x, y \in M$. We consider a chain α of geodesic segments consisting of

(*a*) a sequence of points $x = x_0, x_1, \ldots, x_k = y$ in M;

(*b*) a sequence of points $a_1, b_1, \ldots, a_k, b_k$ in I;

(*c*) projective maps $f_1, \ldots, f_k$ such that $f_i(a_i) = x_{i-1}$ $f_i(b_i) = x_i$ $\forall i = 1, \ldots, k$ and put

$$L(\alpha) = \sum_i \omega(a_i, b_i)\,.$$

Then the pseudodistance p between x and y is given by

$$(1.8) \qquad p(x, y) = \inf L(\alpha)$$

where the infimum is taken over all chains α as above.

The following properties of p are all of immediate proof:

(a) If $f : \mathrm{I} \to \mathrm{M}$ is projective, then $p(f(a), f(b)) \leq \omega(a, b)$ $\forall a, b \in \mathrm{I}$;

(b) If δ is a pseudodistance on M with the property (a), then $\delta(x, y) \leq \leq p(x, y)$ $\forall x, y \in \mathrm{M}$.

The following theorem is due to Wu ([12]).

THEOREM 1.1. *The pseudodistance p is the integrated form of the pseudometric* P, i.e.

$$\forall x, y \in \mathrm{M} \qquad p(x, y) = \inf_{\gamma} \int_{\gamma} \mathrm{P} \tag{1.9}$$

where the infimum is taken over all C^∞-curves γ in M *joining x and y.*

Following Rinow ([9]), Theorem 1.1 implies that p is an inner pseudodistance.

In the following section we want to establish an explicit expression of the pseudometric P for a large class of manifolds.

§ 2. THE PROJECTIVE PSEUDOMETRIC P AND PROJECTIVE TRANSFORMATIONS

We consider the class $\mathcal{G}$ of C^∞-manifolds with a symmetric connection Γ such that

(a) the symmetric part of the Ricci tensor is parallel and negative semidefinite;

(b) Γ is complete.

Note that in the equation of Schwarz (1.4) the Ricci tensor can be replaced by its symmetric part. If $\mathrm{M} \in \mathcal{G}$ the right term of (1.4) is constant and non positive. Indeed, with obvious notations, given a geodesic γ with affine parameter t

$$\nabla_{\dot\gamma} \mathrm{R}(\dot\gamma) = \mathrm{R}(\nabla_{\dot\gamma} \dot\gamma) + (\nabla_{\dot\gamma} \mathrm{R})(\dot\gamma) = 0 .$$

Moreover equation (1.4) can be integrated. Given $x = \gamma(0)$ and

$$-\frac{1}{2} k^2 = \frac{2}{n-1} \sum_{i,j} \mathrm{R}_{ij}(x) \frac{d\gamma^i}{dt}(0) \frac{d\gamma^j}{dt}(0) \tag{2.1}$$

there exist real numbers α, β, δ with $\alpha, \beta \neq 0$ and real numbers a, b, c, d such that

$$p(t) = \alpha [1 - \beta \exp(k t)]^{-1} + \delta \qquad \text{if } k > 0 \tag{2.2}$$

and

$$p(t) = (at + b)(ct + d)^{-1} \qquad \text{if } k = 0 . \tag{2.3}$$

We can now prove the following:

THEOREM 2.1. *Let* M $\in \mathcal{G}$ *and let* R *be the relative Ricci tensor. Then*

$$\forall x \in M, \ X \in TM_x \qquad P(x, X) = (n-1)^{-1/2} \mid R(x)(X, X) \mid^{1/2} \tag{2.3}$$

Proof. *a*) Let $x \in M$ and $X \in TM_X$ and suppose at first that

$$R(x)(X, X) < 0 . \tag{2.4}$$

Let $f: I \rightarrow M$ be any projective map such that $f(0) = x$ and $df_0(V) = X$, with $V \in \mathbf{R}$ and denote with γ the corresponding geodesic, with t as affine parameter running over an interval $J \subset \mathbf{R}$. Shrinking the interval J, if necessary, the affine parameter t can be expressed in terms of the projective one so that we can suppose to have the representation

$$\forall u \in I \qquad f(u) = \gamma(t(u)) . \tag{2.5}$$

Since a translation of the affine parameter doesn't affect the projective parameter, we can suppose that $t(0) = 0$. Hence if we put

$$-(n-1)\frac{1}{4}k^2 = \sum_{i,j} R_{ij}(\gamma(t))\frac{d\gamma^i}{dt}(t)\frac{d\gamma^j}{dt}(t) \tag{2.5}$$

from the general expression (2.2) and from (2.5) we have

$$\forall t \in J \quad u(t) = b[1 - a\exp(kt)]^{-1} - b(1-a)^{-1} \qquad a, b \in \mathbf{R}^* \tag{2.6}$$

Hence

$$\begin{aligned} P(x, X) &= \inf\{\mid V\mid_0 \mid f: I \rightarrow M \text{ proj. } f(0) = x, \dot\gamma(0)\frac{dt}{du}(0)V = X\} = \\ &= \inf\left\{ \left| \frac{du}{dt}(0) \right| \frac{\|X\|}{\|\dot\gamma(0)\|} \,\middle|\, \gamma: J \rightarrow M \quad \text{geodesic}, \quad \gamma(0) = x, u(t) \right. \\ &\qquad \text{proj. par., } u(0) = 0, u(J) \supseteq (-1, 1)\Big\} \end{aligned} \tag{2.7}$$

where $\|\,.\,\|$ is any norm on TM_x. Now we show that we can assume $\dot\gamma(0) = X$. Choosing any $a \in \mathbf{R}^*$ and setting $t^* = at$, $t^* \in J^* = \{at \mid t \in J\}$ and

$$\gamma^*(t^*) = \gamma\left(\frac{t^*}{a}\right), \quad t^* \in J^* \tag{2.8}$$

we have with obvious notations

$$\{u\,,t\} = [\,\{u\,,t\} - \{u^*\,,t\}\,]\left(\frac{\mathrm{d}t}{\mathrm{d}t^*}\right)^2 = \frac{1}{a^2}\{u\,,t\} =$$

$$= \frac{2}{n-1}\,\mathrm{R}\,(\gamma\,(t))\,(\dot\gamma\,(t)\,,\dot\gamma\,(t))\cdot\frac{1}{a^2} = \frac{2}{n-1}\,\frac{1}{a^2}\,\mathrm{R}\,(x)\,(\mathrm{X}\,,\mathrm{X}) = \tag{2.9}$$

$$\frac{2}{n-1}\,\mathrm{R}\,(x)\,(\dot\gamma^*\,(0)\,,\dot\gamma^*\,(0)) = \frac{2}{n-1}\,\mathrm{R}\,(\gamma^*\,(t^*))\,(\dot\gamma^*\,(t^*)\,,\dot\gamma^*\,(t^*)) =$$

$$= \{u^*\,,t^*\}\,.$$

Hence a projective parameter u^* relative to t^* is given by

$$u^*\,(t^*) = u\left(\frac{t^*}{a}\right) \tag{2.10}$$

and therefore

$$\left|\frac{\mathrm{d}u^*}{\mathrm{d}t^*}\right|_{t^*=0}\left|\frac{\|\mathrm{X}\|}{\|\dot\gamma^*\,(0)\|}\right. = \left|\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{t=0}\left|\frac{\|\mathrm{X}\|}{\|\dot\gamma\,(0)\|}\right. \tag{2.11}$$

We have only to choose $\alpha\in\mathrm{R}^*$ such that $\dot\gamma^*\,(0) = \mathrm{X}$ to obtain that

$$\mathrm{P}\,(x\,,\mathrm{X}) = \inf\left\{\left|\frac{du}{dt}\right|_{t=0}\;\middle|\;\begin{array}{l}\gamma:(-\varepsilon\,,\varepsilon)\to\mathrm{M}\,(\varepsilon>0)\text{ is a geodesic,}\\ \gamma\,(0)=x\,,\ \dot\gamma\,(0)=\mathrm{X}\ \ u\,(t)\text{ proj. par.}\\ \text{with } u\,(0)=0\,,\ u\,(-\varepsilon\,,\varepsilon)\supseteq(-1\,,1)\end{array}\right\} \tag{2.11}$$

From (2.7) we get immediately that

$$\mathrm{P}\,(x\,,\mathrm{X}) = \inf_{(a,b)\in\mathrm{C}}\,\{k\,|\,b\,a\,|\,(1-a)^{-2}\} \tag{2.12}$$

where

$$\mathrm{C} = \left\{(a\,,b)\in\mathbf{R}^2\;\middle|\;\begin{array}{l}\exists\text{ interval }\mathrm{J}\subset\mathbf{R}\ \ 0\in\mathrm{J}\text{ such that the}\\ \text{given in (2.6) maps J onto I}\end{array}\right\}$$

Since

$$\mathrm{C} = \left\{(a\,,b)\in\mathbf{R}^2\;\middle|\;\begin{array}{ll}\text{i)} & a<0\text{ and }|\,b\,|\geq 1-a\quad|\,b\,a\,|\geq 1-a\\ \text{ii)} & a\in(0\,,1)\text{ and }a\,|\,b\,|\geq 1-a\\ \text{iii)} & a\in(1\,,\infty)\text{ and }|\,b\,|\geq 1-a\end{array}\right\}$$

and since

$$\inf_{\mathrm{C}}\,\{|\,b\,a\,|\,(1-a)^{-2}\,k\} = \frac{1}{2}\,k \tag{2.13}$$

we have from (2.12) and (2.13)

$$\text{(2.14)} \qquad P(x, X) = \frac{1}{2} k = (n-1)^{-1/2} \mid R(x)(X, X) \mid^{1/2}.$$

b) If $\qquad R(x)(X, X) = 0$

denote with $\gamma : \mathbf{R} \to M$ the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = X$. Setting now

$$\text{(2.15)} \qquad J_n = \left(-\frac{n^2}{2n-1}, n^2\right) \quad \text{and } u_n : J_n \to (-1, 1)$$

$$u_n(t) = n\, t\, [(n-1)\, t + n^2]^{-1} \qquad \forall t \in J_n$$

denote with $t_n(u)$ the inverse map of u_n and consider $f_n : I \to M$ given by

$$f_n(u) = \gamma(t_n(u)) \qquad \forall u \in I.$$

They are projective parameters and $f_n(0) = n\, X$, so that

$$\text{(2.16)} \qquad P(x, X) = 0.$$

Our conclusion follows from (2.14) and (2.16). Q.E.D

The above theorem extends a theorem due to Kobayashi-Sasaki ([6]) about Einstein complete manifolds. The preceding result will now be applied to the study of the group of projective transformations Proj (M) of a manifold M belonging to $\mathcal{G}$. Recall that a diffeomorphism f of a manifold M endowed with covariant differentiation ∇ is said to be projective if the connection defined by $f^* \nabla$ is projectively equivalent to that defined by ∇. If $f^*\nabla = \nabla$, the f is said to be affine. It is easy to see that every projective transformation maps projective maps into projective maps so that we have that

$$\text{(2.17)} \qquad \forall f \in \text{Proj}(M) \qquad P(x, X) = P(f(x), f_* X) \qquad \forall x \in M\ \forall X \in TM_x$$

We can now give a new proof of a classical theorem due to Nagano ([8]):

THEOREM 2.2. *Let* M *be a manifold belonging to* $\mathcal{G}$. *Then every projective transformation is affine,* i.e. Proj (M) = Aff (M).

Proof. Let $f \in$ Proj (M) and R, R* be the Ricci tensors relative to the connections defined by ∇ and $f^*\nabla$ respectively, say Γ and Γ^*. Then the local expression of Γ and Γ^* is given by (1.1) for some global 1-form ϕ. It is easy to see that

$$\text{(2.18)} \qquad R^*(x)(Y, Z) = R(f(x))(f_* Y, f_* Z) \qquad \forall x \in M, Y, Z \in TM_x$$

and so, thanks to (2.17), (2.18) and Theorem 2.1 we have that $R = R^*$. Therefore from (1.2) we get

$$\phi_{ij} = n\,\phi_{ji} \qquad \forall i, j = 1, \ldots, n \tag{2.19}$$

and so, interchanging the role of i and j, we have that

$$\phi_{ij} = 0 .$$

Let now $u : \mathbf{R} \to M$ be any geodesic relative to the connection Γ, with affine parameter $t \in \mathbf{R}$. Put

$$\sigma(t) = \phi_{u(t)}(\dot{u}(t)) \qquad \forall t \in \mathbf{R} . \tag{2.21}$$

Using (2.20) a simple calculation shows that

$$\forall t \in \mathbf{R} \quad \frac{d\sigma}{dt}(t) = \sum_{j,k} \nabla_j \phi_k(u(t)) \frac{du^j}{dt}(t) \frac{du^k}{dt}(t) = (\sigma(t))^2 \tag{2.22}$$

Since there are no global C^1-solutions on $\mathbf{R}$ of the equation (2.22), which are not identically zero, we have $\phi_{u(0)}(\dot{u}(0)) = 0$; because $u(0)$ and $\dot{u}(0)$ are arbitrary, we obtain $\phi = 0$ and so f is an affinity. Q.E.D.

§ 3. Open convex regular cones

Throughout this section Ω will be a subset of $\mathbf{R}^n$ such that

a) $\forall t \in \mathbf{R}_+^* \ \forall x \in \Omega \ tx \in \Omega$;

b) Ω is regular, i.e. contains no affine line;

c) Ω is open and convex.

The dual Ω^* of Ω defined by

$$\Omega^* = \{x^* \in \mathbf{R}^{n*} \mid < x , x^* > > 0 \qquad \forall x \in \bar{\Omega} \setminus \{0\}\} \tag{3.1}$$

is an open convex regular cone of $\mathbf{R}^{n*}$. Let dx be the Lebesgue measure on $\mathbf{R}^n$: the characteristic function ϕ of Ω, defined by the absolutely convergent integral

$$\phi(x) = \int_{\Omega} \exp(- < x , x^* >)\, dx^* \qquad (x \in \Omega) \tag{3.2}$$

is a C^∞-function on Ω and $\log \phi$ is strongly convex. We can therefore define, following Vinberg ([11]), a Riemannian metric g on Ω, called characteristic metric, whose components are given by

$$(3.3) \qquad \forall x \in \Omega \qquad g_{ij}(x) = \frac{\partial^2 \log \phi}{\partial x^i \partial x^j}(x)$$

It is easy to see that the group of automorphism of Ω, i.e. $\mathrm{Aut}(\Omega) = \{A \in GL(n, \mathbf{R}) \mid A(\Omega) = \Omega\}$ acts on Ω as a group of isometries for the Riemannian manifold (Ω, g). We say that Ω is self-adjoint if there exists some scalar product $(\ ,\ )$ on $\mathbf{R}^n$ such that $y \in \Omega$ if and only if

$$(x, y) > 0 \ \forall y \in \bar{\Omega} \setminus \{0\}.$$

If a cone is self-adjoint, we have that

$$(3.4) \qquad \phi(x) = \int_\Omega \exp[-(x, y)]\, dy$$

and that Ω and Ω^* are linearly isomorphic: under this identification it is possible to define an involutorial isometry $*$ on Ω by means of

$$(3.5) \qquad x^* = -d \log \phi(x).$$

If the cone is affinely-homogeneous (as we will suppose from now on) the involution $*$ has an unique fixed point, called p: in this case Ω has a natural structure of symmetric space. The cone Ω is said to be reducible if there is a decomposition $\mathbf{R}^n = \mathbf{R}^p \times \mathbf{R}^q$ (with $p, q \neq 0$ and $p + q = n$) and two open convex regular cones Ω_1 in $\mathbf{R}^p$ and Ω_2 in $\mathbf{R}^q$ such that $\Omega = \Omega_1 \times \Omega_2$.

The following theorem is due to Rothaus ([10]).

THEOREM 3.1. *If Ω is self-adjoint and affinely-homogeneous, then*

a) (Ω, g) *is a complete Riemannian manifold with non-positive sectional curvature*

b) There exists a coordinate system in $\mathbf{R}^n$ in which at the point p the metric tensor is given by the identity and the Ricci tensor is given by $R_p = \mathrm{diag}(0, d_1, \dots, d_{n-1})$ *with* $d_i \leq 0 \ \forall i = 1, \dots, n-1$. *Moreover if Ω is irreducible, then* $d_1 = d_2 = \dots = d_{n-1} < 0$.

We now introduce a foliation in Ω by means of

$$(3.6) \qquad \forall \lambda \in R^*_+ \qquad F_\lambda = \{x \in \Omega \mid \phi(x) = \lambda\}$$

and prove the following:

THEOREM 3.2. *(a) Each F_λ endowed with the induced Riemannian metric is a complete Riemannian manifold and is a maximal integral submanifold relative o the distribution* $L_x = \{t\, x \mid t \in \mathbf{R}\}$ *(under the usual identification* $T\Omega_x = \mathbf{R}^n$*)*.

(*b*) *$\forall \lambda \in \mathbf{R}_+^*$ the map $\psi_\lambda : \Omega \to F \times \mathbf{R}_+^*$*

$$\psi_\lambda(x) = \left(\left[\frac{\phi(x)}{\lambda}\right]^{1/n}, \left[\frac{\phi(x)}{\lambda}\right]^{1/n} x\right) \quad (x \in \Omega) \tag{3.7}$$

is a diffeomorphism and is an isometry when we provide $\mathbf{R}_+^*$ with the metric

$$ds^2 = \mathbf{n} \cdot \frac{dx^2}{x^2}$$

(*c*) *When Ω is irreducible, F_λ is Einstein with negative scalar curvature and the Ricci tensor S of F_λ is the restriction of the Ricci tensor R of Ω.*

Proof. (*a*) It is enough to prove that for every $x \in \Omega$ $TF_{\lambda,x} \perp L_x$, that is $\sum_{i,i} g_{ij}(x)\, x^i v^j = 0$ $\forall v$ such that $\sum \frac{\partial \phi}{\partial x^j}(x)\, v^j = 0$. This follows from the fact that the characteristic function is homogeneous of degree $-n$ and so, by Euler' theorem, we have

$$\forall j = 1, \ldots, n \sum \frac{\partial \log \phi}{\partial x^i x^j}(x)\, x^i = -\frac{\partial \log \phi}{\partial x^j}(x) \tag{3.8}$$

(*b*) The proof of this part is a simple calculation and we leave it out.

(*c*) We first observe that if (Γ^i_{jk}) are the Christoffel symbols, the homogeneity implies that the components (R^i_{jkl}) of the curvature tensor and those of the Ricci tensor (R_{ij}) are given by

$$R^i_{jkl} = \frac{1}{2}\left(\frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l}\right) \tag{3.9}$$

$$R_{ij} = -\frac{1}{2} g_{ij} + \frac{1}{2} \sum_k \frac{\partial \Gamma^k_{ij}}{\partial x^k} \tag{3.10}$$

Since we have that

$$\sum_{i,i} R_{ij}(x)\, x^i x^j = 0 \qquad \forall x \in \Omega \tag{3.11}$$

and since by point (*a*) each F_λ is an integral manifold relative to $\{L_x\}_{x \in \Omega}$ our assertion follows from Theorem (3.1) and from (3.11). Q.E.D.

We prove now the main result of this section:

THEOREM 3.3. *Let the cone Ω be self-adjoint, affinely-homogeneous and irreducible. Then $\forall x, y \in \Omega$*

$$p(x, y) = 0$$

if and only if $x = t\, y$ for some $t \in \mathbf{R}_+^*$.

Proof. *a*) Let us suppose that $x = t_0 y$ ($t_0 \in \mathbf{R}_+^*$) and put $\gamma_t = t x + (1 - t) y$ $t \in [0, 1]$. Then the vectors $(x - y)$ and γ_t are on the same straight line through 0 and so by (3.11)

$$R(\gamma_t)(\dot\gamma_t, \dot\gamma_t) = 0 \qquad \forall t \in [0, 1]. \tag{3.12}$$

By Theorem 2.1 we have that $P(\gamma_t, \dot\gamma_t) = 0$ identically and so since p is the integrated form of P, we have $p(x, y) = 0$.

b) If $p(x, y) = 0$ for some $x \neq y \in \Omega$, there is a sequence of C^∞-curves $\gamma_n : [0, 1] \to \Omega$ joining x and y with

$$\lim_n \int_\gamma P = 0 \tag{3.13}$$

Let F_λ be the leaf through the point x: since homotheties of the cone are isometries, we can suppose that $\lambda = 1$. For $z = (\phi(y))^{1/n} y \in F_1$ we have

$$p(x, z) \leq p(x, y) + p(y, z) = 0 \tag{3.14}$$

since $p(x, y) = 0$ and $p(y, z) =$; by *a*). We define at this point

$$\gamma_n^*(t) = \phi(\gamma_n(t))^{1/n} \gamma_n(t) \qquad \forall t \in [0, 1] \tag{3.15}$$

so that γ_n^* are C^∞-curves in F_1 joining x and z. Because for every $x \in \Omega$ and t in $\mathbf{R}_+^*$ $t^2 R(t x) = R(x)$ and setting

$$\psi_n(t) = \frac{1}{n} \phi(\gamma_n(t))^{(1/n)-1} \sum_i \frac{\partial \phi}{\partial x^1}(\gamma_n(t)) \frac{d\gamma_n^i}{dt}(t)$$

we have

$$\begin{aligned} R(\gamma_n^*(t))(\dot\gamma_n^*(t), \dot\gamma_n^*(t)) &= \phi(\gamma_n(t))^{2/n} \psi_n(t)^2 R(\gamma_n(t))(\gamma_n(t), \gamma_n(t)) + \\ &+ 2(\phi_n(t))^{-2/n} R(\gamma_n(t))(\psi_n(t)\gamma_n(t), \phi(\gamma_n(t))^{1/n} \dot\gamma_n(t)) + \\ &+ R(\gamma_n(t))(\dot\gamma_n(t), \dot\gamma_n(t)). \end{aligned} \tag{3.16}$$

Since the cone is irreducible, by Theorems (3.1) (*c*) at each point $x \in \Omega$ the Ricci tensor vanishes along an unique direction, namely the one generated by x; so from (3.16) we have

$$R(\gamma_n^*)(\dot\gamma_n^*, \dot\gamma_n^*) = R(\gamma_n)(\dot\gamma_n, \dot\gamma_n). \tag{3.17}$$

By Theorem (3.1) (*b*) the Ricci tensor of each leaf is a negative multiple of the metric tensor (say $S = dg$, $d < 0$) so that

$$\int_{\gamma} P(\gamma_n, \dot{\gamma}_n) = \int_{\gamma_n^*} P(\gamma_n^*, \dot{\gamma}_n^*) = \left[\frac{|d|}{n-1}\right]^{1/2} \int_0^1 \| \dot{\gamma}_n^* \|_{\gamma_n^*(t)} \, dt \tag{3.18}$$

By (3.13) and (3.18) we have that the Riemannian distance d_Ω between x and z is zero, and so $x = z$, that is $x = (\phi(y))^{1/n} y$. Q.E.D.

THEOREM 3.4. *Let the cone* Ω *with dimension* $n \geq 3$ *be as in* Theorem 3.3. *If* K *denotes the scalar curvature then* $\forall x, y \in \Omega$

$$p(x, y) = \frac{1}{n-1} |K|^{1/2} d_\Omega \left(x, \left[\frac{\phi(y)}{\phi(x)}\right]^{1/n} y\right).$$

Proof. Thanks to part *a*) of the proof of Theorem 3.3 we know that

$$\forall x, y \in \Omega \qquad p(x, y) = p\left(x, \left[\frac{\phi(y)}{\phi(x)}\right]^{1/n} y\right) \tag{3.19}$$

so that x and $z = \left[\frac{\phi(y)}{\phi(x)}\right]^{1/n} y$ lie both on the same leaf, say W. By Theorem 3.1 (*c*) W is an Einstein space with Ricci tensor S and so, by Theorem 2.1, we have

$$\forall y \in W \ \forall Y \in TW_y \qquad P_W(y, Y) = \left[\frac{|S(y)(Y, Y)|}{n-2}\right]^{1/2} \tag{3.20}$$

Through the proof of Theorem 3.3 we have proved that

$$\begin{aligned} \forall x, z \in W \quad p(x, z) &= \inf_{\gamma} \int_{\gamma} P = \left[\frac{n-2}{n-1}\right]^{1/2} \inf_{\gamma^*} \int_{\gamma^*} P_W = \\ &= \left[\frac{n-2}{n-1}\right]^{1/2} p_W(x, z) \end{aligned} \tag{3.21}$$

with obvious notations. By Theorem 3.1 (*b*) and (*c*) we have

$$p_W(x, z) = \left[\frac{|d|}{n-2}\right]^{1/2} d_W(x, z) = \left[\frac{|d|}{n-2}\right]^{1/2} d_\Omega(x, z) \tag{3.22}$$

where d_W denotes the distance induced on W by g. Since the scalar curvatur K is given by $K = d(n-1)$ and (3.22) lead to our statement. Q.E.D

In case of reducibility of the cone, we decompose Ω as $\oplus_i^k \Omega_i$ where Ω_i are irreducible cones lying in $\mathbf{R}^{n_i}$ (see e.g. Gentili ([4])): since the Ricci tensor has a matricial representation as direct sum of the Ricci tensors R_i of Ω_i, we obtain that:

$$(3.23) \quad \forall\, x \in \Omega\, \forall\, X \in T\Omega_x \quad P(x, X) = \frac{1}{(n-1)^{1/2}} \sum_{=1}^{k} [(n_i - 1)(P_i(x_i, X_i))^2]^{1/2}$$

where $x = (x_1, \ldots, x_k)$ and $X = (X_1, \ldots, X_k)$. A simple application of Theorem 3.3. leads to the following

COROLLARY 3.1. *Let Ω be self-adjoint, homogeneous and reducible as $\oplus^k \Omega_i$ where Ω_i are irreducible cones lying in $\mathbf{R}^{n_i}$ with $n_i \geq 3$. If $p(x, y) = 0$ for some $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$ then there exist $t_i \in \mathbf{R}_+^*$ $(i = 1, \ldots, k)$ such that $x_i = t_i y_i \quad \forall i = 1, \ldots, k$.*

Theorem 3.4 shows that the projective pseudodistance p on an irreducible cone is completely determined by its restriction to a single leaf, vanishing on any line through the origin.

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