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A completion of A. Bressan's work on axiomatic foundations of the Mach Painlevé type for various classical theories of continuous media. Part 1. Completion of Bressan's work based on the notion of gravitational equivalence of affine inertial frames

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Meccanica dei continui. — *A completion of A. Bressan's work on axiomatic foundations of the Mach Painlevé type for various classical theories of continuous media.* Part 1. *Completion of Bressan's work based on the notion of gravitational equivalence of affine inertial frames.* Nota di ADRIANO MONTANARO, presentata (*) dal Corrisp. A. BRESSAN.

ABSTRACT. — The work [3], where various classical theories on continuous bodies are axiomatized from the Mach-Painlevè point of view, is completed here in two alternative ways; in that work, among other things, affine inertial frames are defined within classical kinematics.

Here, in Part I, a thermodynamic theory of continuous bodies, in which electrostatic phenomena are not excluded, is dealt with. The notion of gravitational equivalence among affine inertial frames and the notion of gravitational isotropy of these frames are introduced; it is shown that the isotropic inertial frames, gravitationally equivalent to a fixed frame of this kind, are those linked to this by a (possibly improper) Galilean transformation. The Euclidean physical metric on inertial spaces is consequently determined, without introducing it as a primitive notion; and this is the main completion of [3] which is obtained here.

KEY WORDS: Axiomatization; Continuum; Thermodynamics.

RIASSUNTO. — *Un completamento del lavoro di A. Bressan sui fondamenti assiomatici alla Mach-Painlevè per varie teorie classiche dei mezzi continui. Parte I. Completamento del lavoro di Bressan basato sulla nozione di equivalenza gravitazionale tra riferimenti inerziali affini.* Si completa in due maniere alternative il lavoro [3] di assiomatizzazione alla Mach Painlevé di varie teorie classiche di sistemi continui, ove, tra l'altro, riguardo alla cinematica classica si arriva a definire i riferimenti inerziali affini.

Nella Parte 1 si considera una generica teoria termodinamica dei sistemi continui nella quale non si escludono fenomeni elettrostatici. Si introducono le nozioni di equivalenza gravitazionale tra riferimenti inerziali affini e di isotropia gravitazionale di tali riferimenti; si dimostra che tutti e soli i riferimenti inerziali isotropi gravitazionalmente equivalenti ad un fissato tale riferimento sono legati a questo da una trasformazione Galileiana (impropria). La metrica fisica Euclidea sugli spazi inerziali risulta quindi determinata, senza bisogno di introdurla come nozione primitiva; e ciò costituisce il principale completamento di [3] qui conseguito.

N. 1. INTRODUCTION (**)

In [3] A. Bressan deals with the axiomatization according to Mach and Painlevé, of any classical theory $\mathscr{T}_{r,s,t}$ for 3-dimensional continuous media,

(*) Nella seduta del 20 giugno 1986.

(**) This work has been performed within the activity sphere of the Consiglio Nazionale delle Ricerche in the academic years 1984-85 and 1985-86.

where diffusion processes, (non-electrostatic) electromagnetic phenomena, and chemical reactions are excluded, and where gravitational, thermodynamic, and electrostatic phenomena are excluded [included] in case r, t or s respectively equals 0 [1]. Especially as a kinematic contribution to this axiomatization, an extension to continuous media is performed in [3] for the axiomatization of classical and relativistic kinematics that is presented in [2] (*i*) from a unified point of view, (ii) by use of very few and purely kinematic primitive concepts, and (iii) referring to particle systems. This axiomatization of kinematics is complete only in the relativistic case in that Minkowski's metric is reached (in it), but in the classical case only affine inertial space-time co-ordinates can be defined. In this case the theory presented in [2] already had a completion. More in detail in the previous work [1], where only particle systems are treated, the physical Euclidean metrics on inertial spaces had been defined, practically on the basis of the same kinematic primitive concepts used in [2] and also a dynamic one.

In the present paper two completions of [3] are presented in Parts 1 and 2 respectively. The first is an analogue for continuous bodies of the aforementioned completion of [2] made in [1], but it differs from the latter in several features. E.g. in connection with $\mathscr{T}_{1,s,t}$ gravitation is the dynamic concept used to reach the goal, and it is defined by means of purely kinematic primitives. In the second completion of [3] only the kinematic primitives introduced in [3], and the notion of physical equivalence between inertial affine frames, are used to reach the goal. Since this approach avoids the use of gravitation, it is fit for extensions to special relativity.

Of course, as in [1] and [3] physical possibility and necessity are used in an essential way.

For more information about the type of axiomatization used in the present work and related topics, see the introduction in [3]. Moreover, this work is presupposed here in that several definitions and theorems in [3] are often simply referred to.

More in detail, in N. 2 an equivalence relation for motions (physically) possible for a body is defined on the basis of the notion of pre-matter points introduced in [3]. N. 2 refers to any theory $\mathscr{T}_{r,s,t}$ $(r, s, t = 0, 1)$.

In N. 3 the notions of gravitationally isotropic inertial (affine) frames and gravitationally equivalent isotropic inertial frames are introduced; the co-ordinate transformations between arbitrary gravitationally isotropic inertial frames are characterized substantially on the basis of only one simple axiom of physical possibility (Ax. 3.4). As a corollary it follows that two arbitrary gravitationally isotropic inertial frames are gravitationally equivalent if and only if their co-ordinate transformation is Galilean. Hence the metric on inertial spaces is determined.

The first two sections of Part 2 - NN. 4-5 - deal with the properties of physical homogeneity and isotropy of space-time and with (physical) equivalence of affine inertial frames in connection with any theory $\mathscr{T}_{r,s,0}$ $(r, s = 0, 1)$. These notions, already considered in [1] in connection with particle systems, are framed here for continuous media in a rather different way; more in detail e.g. the analogue for continuous media of Ax. 28.1, p. 177 in [1], is a theorem here on the basis of a deterministic axiom, Ax. 5.1, and an existence axiom, Ax. 5.2.

Thus the physical Euclidean metrics on inertial spaces (relative to a given length unit) are defined (in N. 4) as well as in [1] like in Part 1. For $\mathscr{T}_{r,s,0}$ this definition depends on some non-kinematic primitive notions, if and only if $s=1$.

Th. 5.2 is the analogue of Th. 3.3 (α) and characterizes the co-ordinate transformations between arbitrary isotropic inertial frames.

In N. 6 some connections between the two completions presented in Parts 1, 2 are emphasized.

N. 2. EQUIVALENT POSSIBLE MOTIONS OF A BODY, REPRESENTATIONS OF MOTIONS, PROCESSES

All this number refers to any theory $\mathscr{T}_{r,s,t}$ $(r, s, t = 0, 1)$. Remember that CAIF denotes the class of classical affine inertial frames – see [3, N. 5] –; and assume, once for all, that

(2.1) $\qquad \mathscr{P} \in \mathscr{M}\mathscr{P}$, $\mathscr{B} = \mathbf{B}_{\mathscr{P}}$, $\varphi \equiv (x_\alpha) \in$ CAIF [(1)], $x_0 = t$ – see [3, Def. 3.2] –.

Furthermore remember that – see remark below Def. 3.2 in [3] – $\mathscr{M}$ is a (physically) possible motion for the body $\mathscr{B}$ in case (i) $\mathscr{M} = \{\mathscr{A}_{\mathscr{Q}}\}_{\mathscr{Q} \leq \mathscr{P}}$ where $\mathscr{A}_{\mathscr{Q}}$ is a mathematical subset of EP for $\mathscr{Q} \leq \mathscr{P}$ and (ii) it is possible that, for every $\mathscr{Q} \leq \mathscr{P}$ $\mathscr{A}_{\mathscr{Q}} = \mathrm{WT}_{\mathscr{Q}}$; furthermore – see Def. 3.2 in [3] which refers to the actual motion of $\mathscr{B}$ – the φ-position $\mathrm{P}_{\varphi,\mathscr{P},\mathscr{M},t}$ and φ-configuration $\mathrm{C}_{\varphi,\mathscr{B},\mathscr{M},t}$ of $\mathscr{P}$ or $\mathscr{B}$ respectively, in the motion $\mathscr{M}$ at the instant t, can be defined by

$$\mathbf{P}_{\varphi,\mathscr{P},\mathscr{M},t} = \mathscr{A}_{\mathscr{P}} \cap \varphi\text{-Inst}_t\,, \qquad \mathbf{C}_{\varphi,\mathscr{B},\mathscr{M},t} = \{\mathbf{P}_{\varphi,\mathscr{Q},\mathscr{M},t}\}_{\mathscr{Q} \in \mathscr{B}}\,. \tag{2.2}$$

I shall use the term *regular domain* in a sense exactly defined, e.g. as the closure of a (finitely disconnected) open set whose boundaries are piece-wise C^2 [(2)].

Ax. 2.1. (Regularity). *Assume that $\mathscr{M}$ is a possible motion for $\mathscr{B}$. If $\mathscr{Q}, \mathscr{R} \in \mathscr{B} \cup \{\varnothing\}$, $\mathscr{Q} \curlywedge \mathscr{R} = \varnothing$, and $t \in \mathbf{R}$, then* (a) $\varphi(\mathbf{P}_{\varphi,\mathscr{Q},\mathscr{M},t})$ *is (either empty or) a regular domain of* $\mathbf{R}^3$, *and* (b) $\partial\varphi(\mathbf{P}_{\varphi,\mathscr{Q},\mathscr{M},t}) \cap \partial\varphi(\mathbf{P}_{\varphi,\mathscr{R},\mathscr{M},t})$ [(3)] *is (either empty or) a regular surface of* $\mathbf{R}^3$.

(1) Greek [Latin] indexes are meant to run from 0 [1] to 3.

(2) See e.g. [4], p. 113, for complete and rigorous definitions, where " regular region " is used instead of " regular domain ".

(3) If D is a regular domain of $\mathbf{R}^3$ then ∂ D and D° denote its boundary and its internal part respectively.

DEF. 2.1. *The possible motions $\mathcal{M}$ and $\mathcal{M}'$ of $\mathcal{B}$ are said to be* MP- *equivalent (i.e. equivalent with respect to the matter points of $\mathcal{B}$) if for each family* $\mathbf{M} = \{\mathcal{P}_n\}_{n \in \mathrm{N}}$ *such that* (i) $\mathcal{P}_n \in \mathcal{M}\mathcal{P}$ *for each* $n \in \mathbf{N}$, (ii) *no* $\mathcal{Q} \in \mathcal{B}$ $(\mathcal{Q} \neq \varnothing\}$ *is a subportion of every* $\mathcal{P}_n$ $(n \in \mathbf{N})$, *and* (iii) $\mathcal{P}_{n+1} \leq \mathcal{P}_n$ $(n \in \mathbf{N})$, *we have that*

(α) $\mathbf{M}$ *is a pre-matter point in the motion* $\mathcal{M}$ – *see* Def. 3.7 (*a*), (*b*) *in* [3] – *if and only if* $\mathbf{M}$ *is a pre-matter point in the motion* $\mathcal{M}'$.

By Def. 3.7 (d) in [3], we obviously have that *the possible motions $\mathcal{M}$ and $\mathcal{M}'$ of $\mathcal{B}$ are* MP – *equivalent if and only if the classes of matter points of $\mathcal{B}$ in $\mathcal{M}$ and $\mathcal{M}'$ coincide.*

In the sequel only motions without rupture, tear and slip surfaces will be considered. Therefore it is natural to assume the following axiom and to fix the attention on the maximal classes of MP-equivalent motions.

AX. 2.2 (Existence). *Assume that* $\varphi \in$ CAIF *and $\mathcal{B}$ is an arbitrary body. Then it is possible for $\mathcal{B}$ to undergo a φ–rest motion $\mathcal{M}^0$, i.e. a motion for which*

$$\mathbf{C}_{\varphi,\mathcal{B},\mathcal{M}^0,t} = \mathbf{C}_{\varphi,\mathcal{B},\mathcal{M}^0,0} \qquad \forall t \in \mathbf{R}\,.$$

Let us consider the (spatial) projection operator π_0 in $\mathbf{R}^4$ defined by $\pi_0(x_0, x_1, x_2, x_3) = (x_1, x_2, x_3)$. In the case above we can call the configuration γ such that $\gamma(\mathcal{Q}) = \mathbf{P}_{\varphi,\mathcal{Q},\mathcal{M}^0,0}$ $\forall\, \mathcal{Q} \in \mathcal{B}$, *a φ-rest configuration* and its representation $\gamma^* = \pi_0 \circ \varphi \circ \gamma$ a *φ-rest φ-configuration*. Fix γ^* once for all, i.e. regard it as a particular *reference configuration*. Furthermore let MP (γ^*) (and MP $(\mathcal{M}^0)$) denote the class of the possible motions for $\mathcal{B}$ MP-equivalent to $\mathcal{M}^0$. In each of them rupture, tear, and slip surfaces are absent. Only possible motions in MP (γ^*) will be considered in the sequel.

Note that if (i) $\mathcal{M}^1 \in$ MP $(\mathcal{M}^0)$, (ii) for some $\psi \in$ CAIF $\delta(\mathcal{Q}) = \mathbf{P}_{\psi,\mathcal{Q},\mathcal{M}^0,0}$ and (iii) $\delta^* = \pi_0 \circ \psi \circ \delta$, then the class MP $(\mathcal{M}^1)$ of the motions MP-equivalent to $\mathcal{M}^1$ obviously equals MP $(\mathcal{M}^0)$ and can be denoted by MP (δ^*) even if δ fails to be a rest configuration.

It is now easy to specify the notion of *connected matter portions*. Let us say that $\mathcal{Q}$ $(\in \mathcal{B})$ is a *γ^*-connected part* of $\mathcal{B}$ if $\gamma^*(\mathcal{Q})$ is a connected subset of $\mathbf{R}^3$, which property is independent of φ. Obviously (a) [(b)] *this case occurs if* $\psi\,(\mathbf{P}_{\psi,\mathcal{Q},\mathcal{M},t})$ *is a connected subset of* $\mathbf{R}^3$ *for some* [*every*] $t \in \mathbf{R}$, *for some* [*every*, $\mathcal{M} \in$ MP (γ^*) *and some* [*every*] $\psi \in$ CAIF.

DEF. 2.3. *Remembering* (2.1), *for* $\mathcal{M} \in$ MP (γ^*)

(α) *the function* $\mathbf{F} : \{\gamma^*(\mathcal{Q})\}_{\mathcal{Q} \leq \mathcal{P}} \times \mathbf{R} \to \mathbf{R}^3$ *defined by* $\mathbf{F}(\gamma^*(\mathcal{Q}), t) = \varphi\,(\mathbf{P}_{\varphi,\mathcal{Q},\mathcal{M},t})$ (*for* $\mathcal{Q} \in \mathcal{B}$) *will be called φ-representation of $\mathcal{M}$ by parts.*

(β) *The function* $f : \gamma^*(\mathcal{P}) \times \mathbf{R} \to \mathbf{R}^3$ *defined by* $f(y, t) = \bigcap\limits_0^\infty \mathbf{F}(\gamma^*(\mathcal{Q}_n), t)$

where (i) $\mathbf{M} = \{\mathcal{Q}_n\}_{n \in \mathrm{N}}$ *is a pre-matter point with* $\mathcal{Q}_0 \in \mathcal{B}$ *and* (ii) $\{y\} = \bigcap\limits_0^\infty \gamma^*(\mathcal{Q}_n)$, *will be called punctual φ-representation of $\mathcal{M}$.*

(γ) $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}, t)$ *is called the* φ*-position of* $\boldsymbol{y}$ *at instant* t (*in* $\mathscr{M}$). *Lastly*

(δ) $\mathscr{M}$ *is said to be* C^2 *at the instant* t *if for each* γ^**-connected* $\mathscr{Q} \in \mathscr{B}$, $\boldsymbol{f}$ *is* C^0 *in* $\gamma^*(\mathscr{Q}) \times \mathbf{R}$ *and it is* C^2 *in* $\gamma^*(\mathscr{Q}) \times \mathrm{I}$ *for some neighbourhood* I *of* t, *open, non-empty, and possibly unbounded.*

Theories of the type $\mathscr{T}_{r,1,t}$ deal with (absolute) temperature and obvious related notions such as e.g.

" θ *is the* (*absolute*) *temperature of the particle* $\boldsymbol{y}$ *at instant* t", and

" ζ *it the* (*actual*) *temperature distribution of the body* $\mathscr{B}$"; hence processes, besides motions, have to be considered in them.

DEF. 2.4 (α) *Remembering* (2.1), *the couple of functions* $\boldsymbol{p} = \langle \boldsymbol{f}, \zeta \rangle$ *where* $\boldsymbol{f} : \gamma^*(\mathscr{P}) \times \mathbf{R} \to \mathbf{R}^3$, *and* $\zeta : \gamma^*(\mathscr{P}) \times \mathbf{R} \to \mathbf{R}^+_*$ [(4)] *is called the* φ*-representation of the* (*actual*) *process of the body* $\mathscr{B} = \mathbf{B}_{\mathscr{P}}$ – *i.e. the process undergone by* $\mathscr{B}$ –, *if* $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}, t)$ *and* $\theta = \zeta(\boldsymbol{y}, t)$, *where* $\boldsymbol{y} \in \gamma^*(\mathscr{P})$, $t \in \mathbf{R}$, *are the* φ*-position and the absolute temperature respectively of* $\boldsymbol{y}$ *at instant* t.

(β) *If* $p = \langle \boldsymbol{f}, \zeta \rangle$ *is the* φ*-representation of the possible process* P *for the body* $\mathscr{B}$ – *briefly* $\mathrm{P} \in \mathrm{PP}_{\mathscr{B}}$ –, *then* $\boldsymbol{f}$ *and* ζ *are called the* φ*-representations of the motion and the temperature distribution, respectively, in* P.

(γ) $\mathrm{P} \in \mathrm{PP}_{\mathscr{B}}$ *is said to be* C^2 *at the instant* t *if both* $\boldsymbol{f}$ *and* ζ *are smooth at the instant* t, *in the sense of* Def. 2.3 (δ).

Sometimes, in the sequel, any φ-representation $\langle \boldsymbol{f}, \zeta \rangle$ will be expressed in accordance with the equalities

$$\begin{aligned} \langle \boldsymbol{f}, \zeta \rangle &= \{\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}, t), \theta = \zeta(\boldsymbol{y}, t)\}_{(\boldsymbol{y},t) \in \gamma^*(\mathscr{P}) \times \mathbf{R}} = \\ &= \{\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}, t), \theta = \zeta(\boldsymbol{y}, t)\} \ldots \end{aligned} \tag{2.3}$$

N. 3. CLASSICAL GRAVITATIONALLY ISOTROPIC AND GRAVITATIONALLY EQUIVALENT (AFFINE) INERTIAL FRAMES

This number is devoted to a theory $\mathscr{T}_{1,s,t}$: by dropping all references to [electromagnetic field], temperature and related notions, one obtains the corresponding results for $[\mathscr{T}_{1,s,t}]\ \mathscr{T}_{1,0,t}$.

The mathematical lemma below is sufficient to deduce the continuity of Newton's gravitational force per unit mass, due to a smooth matter distribution.

LEMMA 3.1. *Assume that* (i) D_1, D_2 *are regular domains of* $\mathbf{R}^3$ *with* D_2 *bounded and* $D_1^0 \cap D_2^0 = \varnothing$ [(5)], (ii) *the function* $\rho(.) : D_2 \to \mathbf{R}$ *is integrable and bounded, and* (iii) $\boldsymbol{x}_1 \in D_1 \cap D_2$; *and set*

$$|\boldsymbol{x} - \boldsymbol{x}'| = \left[\sum_{i=1}^{3} (x_i - x_i')^2\right]^{1/2}; \tag{3.1}$$

(4) $\mathbf{R}^+_* = \{x \in \mathbf{R} \mid x > 0\}$.

(5) See footnote (2).

then

(α) *the improper integral* $\int_{D_2} \rho(\boldsymbol{x}_2)(\boldsymbol{x}_2 - \boldsymbol{x}_1)/|\boldsymbol{x}_2 - \boldsymbol{x}_1|^3 \, dv(\boldsymbol{x}_2)$ *is convergent and*

(β) *the function* $\boldsymbol{g}(.) : D_1 \to \mathbf{R}^3$ *defined by*

$$\boldsymbol{g}(\boldsymbol{x}_1) = \int_{D_2} \rho(\boldsymbol{x}_2)(\boldsymbol{x}_2 - \boldsymbol{x}_1)/|\boldsymbol{x}_2 - \boldsymbol{x}_1|^3 \, dv(\boldsymbol{x}_2) \tag{3.2}$$

it continuous (also when $D_1 \cap D_2 \neq \varnothing$*).*

Lemma 3.1 is well known – see e.g. [4] –; observe only that to prove it one must assume that $\rho(.)$ is integrable and bounded. Hence, for simplicity reasons, processes are assumed to be regular enough to be compatible with the following reasonable axiom.

Ax. 3.1. *Assume* (2.1) *and that* (i) *the body* $\mathscr{B}$ *is at the finite – see* [3, Def. 3.3.] – *and* (ii) $\mu = \mu_{\varphi,t}$ *is the measure satisfying* (7.1) *in* [3]. *Then* μ *is absolutely continuous and its volume density* $\rho(. , t)$ *is bounded.*

Henceforth only bodies occupying bounded configurations will be considered, hence the following is postulated.

Ax. 3.2. *Assume* (2.1) *and let* $\mathscr{M}$ *be a possible motion for* $\mathscr{B} = \mathbf{B}_{\mathscr{P}}$*. Then for every* $t \in \mathbf{R}$ *there exists a* $k > 0$ *such that*

$$\mathscr{P}_t \subset S_{\varphi,k,\mathbf{0}}, \qquad \textit{where}$$

$$\mathscr{Q}_t = \varphi(\mathbf{P}_{\varphi,\mathscr{Q},\mathscr{M},t}) \ \textit{for} \ \mathscr{Q} \in \mathscr{M}\mathscr{P}, \ \textit{and} \tag{3.3}$$

$$S_{\varphi,k,\boldsymbol{c}} = \{\boldsymbol{x} \in \mathbf{R}^3 : (\boldsymbol{x} - \boldsymbol{c})^* (\boldsymbol{x} - \boldsymbol{c}) \leq k^2\} \ (6).$$

Def. 3.1. *Assume that* (i) (2.1) *holds,* (ii) $h > 0$, $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$, $\mathscr{B}_1 \cap \mathscr{B}_2 = \varnothing$, $\mathscr{B}_i = \mathbf{B}_{\mathscr{P}_i}$, $\mathscr{P}_i \in \mathscr{M}\mathscr{P}$ $(i = 1, 2)$, (iii) $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}, t)$ *where* $\boldsymbol{f}$ *is the* φ*-representation of a possible* C^2 *motion for* $\mathscr{B}$, *and* (iv) $\rho = \rho(\boldsymbol{x}, t)$ *is the mass density at* $\boldsymbol{x}, t$ – *see* Ax. 3.1 –. *Then*

$$h \int_{\mathscr{P}_{1,t}} \rho(\boldsymbol{x}, t)\, \boldsymbol{g}(\boldsymbol{x}, \mathscr{P}_{2,t}) \, dv \qquad [h\, \rho(\boldsymbol{x}, t)\, \boldsymbol{g}(\boldsymbol{x}, \mathscr{P}_{2,t})] \ \text{– see (3.2) –}$$

(6) If $\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{R}^3$ then $\mathbf{v}^*$ denotes the transpose of $\mathbf{v}$, and $\mathbf{v}^* \mathbf{v} = \sum_{i=1}^{3} v_i v_i$; analogously, if $\mathbf{A}$, $\mathbf{B}$ are matrices, then $\mathbf{A}^*$ denotes the transpose of A and $\mathbf{A}\,\mathbf{B}$ is the usual product of A with $\mathbf{B}$.

is called the φ*-representation of the* h – *Newtonian gravitational force acted by* $\mathscr{B}_2$ *on* $\mathscr{B}_1$ [*at* $\boldsymbol{x}$ *per unit volume and*] *at time* t.

DEF. 3.2 (in $\mathscr{T}_{1,s,0}$ [in $\mathscr{T}_{1,s,1}$]). *Assume that* $\varphi \equiv (x_\alpha) \in$ CAIF, *and there exists a* $h_\varphi > 0$ *such that for each body* $\mathscr{B} = \mathbf{B}_{\mathscr{P}}$, *each motion* $\mathscr{M}$ *that* $\mathscr{B}$ *can undergo in isolation* [*in the absence of any electromagnetic field*], *and each* $\mathscr{Q} \in \mathscr{B}$, *the following conditions* (α) *and* (β) *are satisfied at every instant* t *in which* $\mathscr{M}$ *is* C^2 *and in which* $\partial \mathscr{Q}_t \cap \partial (\mathscr{P}_t - \mathscr{Q}_t) = \varnothing$.

(α) (*Linear momentum law*)
$$\int_{\mathscr{Q}_t} \ddot{\boldsymbol{x}}\, d\, m = h_\varphi \int_{\mathscr{Q}_t} \boldsymbol{g}(\boldsymbol{x}, \mathscr{P}_t - \mathscr{Q}_t)\, \mathrm{dm};$$

(β) (*Angular momentum law*)
$$\int_{\mathscr{Q}_t} (\boldsymbol{x} - \boldsymbol{o}) \times \ddot{\boldsymbol{x}}\, \mathrm{dm} = h_\varphi \int_{\mathscr{Q}_t} (\boldsymbol{x} - \boldsymbol{o}) \times$$
$$\times\, \boldsymbol{g}(\boldsymbol{x}, \mathscr{P}_t - \mathscr{Q}_t)\, \mathrm{dm};$$

where $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}, t)$, $\boldsymbol{f}$ *is the* φ*-representation of* $\mathscr{M}$, $\ddot{\boldsymbol{x}} = \partial^2 \boldsymbol{f}(\boldsymbol{y}, t)/\partial t^2$, $\boldsymbol{g}(\ldots)$ *is defined as in* (3.2), $\boldsymbol{o}$ *is a fixed point* (*with respect to* φ), *and* $\mathrm{dm} = \rho\, \mathrm{d}\, v$ *is the element of mass at* $\boldsymbol{x}$, t.

Then φ *is said to be a classical gravitationally isotropic* (*affine*) *inertial frame* ($\varphi \in$ CGIIF).

AX. 3.3 (Existence). *There exists some* $\varphi \in$ CGIIF.

AX. 3.4. (Physical possibility in $\mathscr{T}_{1,s,0}$ [in $\mathscr{T}_{1,s,1}$]). *Given* $\varphi \equiv (x_\alpha) \in$ CAIF, $t \in \mathbf{R}$ *and* $(\boldsymbol{c}_1, \boldsymbol{c}_2, r_1, r_2) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}_*^+ \times \mathbf{R}_*^+$ *with* $|\boldsymbol{c}_1 - \boldsymbol{c}_2| > r_1 + r_2$, *there are disjoint* $\mathscr{P}_1, \mathscr{P}_2 \in \mathscr{M}\mathscr{P}$ *and a motion* $\mathscr{M}$ *for the body* $\mathscr{B} = \mathbf{B}_{\mathscr{P}_1 \curlyvee \mathscr{P}_2}$ *such that, at instant* t, (i) $\mathscr{M}$ *is* C^2, (ii) $\mathscr{P}_{i,t} \subseteq S_{\varphi, r_i, \boldsymbol{c}_i}$ ($i = 1, 2$) – see (3.3) –, *and* (iii) *the body* $\mathscr{B}$ *can undergo* $\mathscr{M}$ *in isolation* [*and in the absence of any electromagnetic field*].

LEMMA 3.2 (in $\mathscr{T}_{1,s,0}$ [in $\mathscr{T}_{1,s,1}$]). *For some body* $\mathscr{B} = \mathbf{B}_{\mathscr{P}}$ *with* $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$, $\mathscr{B}_1 \cap \mathscr{B}_2 = \varnothing$, *it is physically possible that* (i) *the acceleration* $\boldsymbol{a}_1$ *of the centre of mass of* $\mathscr{B}_1$ *is* $\neq \mathbf{0}$, *and* (ii) $\mathscr{B}$ *is isolated* [*and in the absence of any electromagnetic field*] (7).

Proof. Assume that $\varphi \equiv (x_\alpha) \in$ CGIIF. Let $(\boldsymbol{c}_1, \boldsymbol{c}_2, r_1, r_2)$, $\mathscr{B} = \mathbf{B}_{\mathscr{P}_1 \curlyvee \mathscr{P}_2}$ and $\mathscr{M}$ satisfy all the conditions in Ax. 3.4 at instant t. Hence $\partial \mathscr{B}_{1,t} \cap \partial \mathscr{B}_{2,t} =$

(7) Lemma 3.2 could be deduced by the following:

Ax. 3.4* (Physical possibility in $\mathscr{T}_{1,s,0}$ [in $\mathscr{T}_{1,s,1}$]) *Given* $\varphi \equiv (x_\alpha) \in$ CGIIF, *there are disjoint* $\mathscr{P}_1, \mathscr{P}_2 \in \mathscr{M}\mathscr{P}$, *and a motion* $\mathscr{M}$ *for* $\mathscr{B} = \mathbf{B}_{\mathscr{P}_1 \curlyvee \mathscr{P}_2}$, *such that for some* $(\boldsymbol{c}_1, \boldsymbol{c}_2, r_1, r_2) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}_*^+ \times \mathbf{R}_*^+$ *with* $|\boldsymbol{c}_1 - \boldsymbol{c}_2| > r_1 + r_2$, *at some instant* t (i) $\mathscr{M}$ *is* C^2, (ii) $\mathscr{P}_{i,t} \subseteq S_{\varphi, r_i, c_i}$ ($i = 1, 2$), *and* (iii) $\mathscr{B}$ *can undergo* $\mathscr{M}$ *in isolation* [*and in the absence of any electromagnetic field*].

However Ax. 3.4 is essential to prove the whole TH. 3.3.

$= \varnothing$, where $\mathscr{B}_i = \mathbf{B}_{\mathscr{P}_i}$, and by Def. 3.2 ($\alpha$) we have

$$\text{(3.4)} \qquad m_1 [\boldsymbol{a}_1(t)]_\varphi = h_\varphi \int_{\mathscr{P}_{1t}} \boldsymbol{g}(\boldsymbol{x}_1, \mathscr{P}_{2,t}) \, \mathrm{dm}_1 ,$$

where

(α) $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}, t)$, $\boldsymbol{f}$ is the φ-representation of the motion $\mathscr{M}$, m_1 is the mass of $\mathscr{B}_1$ and $\boldsymbol{a}_1(t)$ is the acceleration of the centre of mass of $\mathscr{B}_1$ at instant t.

Assume that

(β) $\boldsymbol{i}_r$ is the unit vector of axis x_r;

choosing $\boldsymbol{c}_1, \boldsymbol{c}_2$ with $\boldsymbol{c}_1 - \boldsymbol{c}_2$ parallel to $\boldsymbol{i}_r$ and $(\boldsymbol{c}_2 - \boldsymbol{c}_1) \cdot \boldsymbol{i}_r > 0$, by projection of (3.4) onto this axis we obtain $m_1 [\boldsymbol{a}_1(t) \cdot \boldsymbol{i}_r]_\varphi = h_\varphi \int_{\mathscr{P}_{1,t}} \int_{\mathscr{P}_{2,t}} (\boldsymbol{x}_2 - \boldsymbol{x}_1) \cdot \boldsymbol{i}_r / |\boldsymbol{x}_2 - \boldsymbol{x}_1|^3 \, \mathrm{dm}_2 \, \mathrm{dm}_1$ – see (3.2) –; by Ax. 3.4 (ii) it results $(\boldsymbol{x}_2 - \boldsymbol{x}_1) \cdot \boldsymbol{i}_r > 0$ for each $(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathscr{P}_{1,t} \times \mathscr{P}_{2,t}$, hence $[\boldsymbol{a}_1(t) \cdot \boldsymbol{i}_r]_\varphi > 0$, i.e. $\boldsymbol{a}_1(t) \neq \mathbf{0}$. q.e.d.

Observe that, if both h_φ and h'_φ satisfy the condition in Def. 3.2 for the frame φ, then by Lemma 3.2 it easily follows that $h_\varphi = h'_\varphi$; h_φ is called φ-*Cavendish's constant.*

DEF. 3.3. *φ and ψ ($\in$ CGIIF) are said to be gravitationally equivalent if $h_\varphi = h_\psi$, where both (φ, h_φ) and (ψ, h_ψ) satisfy the conditions* (α) *and* (β) *in* (φ, h_φ), *written within* Def. 3.2.

By Th. 5.3 (α) in [3], if $\varphi \equiv (x_\alpha)$ and $\psi \equiv (z_\alpha) \in$ CAIF, then the transformation $\psi \circ \varphi^{-1}$ is of the type

$$\text{(3.5)} \qquad \boldsymbol{z} = \mathbf{A}\,\boldsymbol{x} - \boldsymbol{b}\,x_0 - \boldsymbol{c}\,, \quad z_0 = \tau\, x_0 - c_0$$

for some $(\mathbf{A}, \boldsymbol{b}, \boldsymbol{c}, \tau, c_0) \in \mathrm{Lin}_* \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^+_* \times \mathbf{R}$,

where

$$\text{(3.6)} \qquad \mathrm{Lin}_* = \{3 \times 3 \text{ real matrices } \mathbf{A} \mid \det \mathbf{A} \neq 0\}\,.$$

If (3.5) represents $\psi \circ \varphi^{-1}$ I express this by setting

$$\text{(3.7)} \qquad \psi = \varphi^{\mathbf{A}\,\boldsymbol{b}\,\boldsymbol{c}}_{\tau\,c_0}\,.$$

The transformation $\psi \circ \varphi^{-1}$ expressed by (3.5) is said to be *proper* [*improper*] *and Galilean* if $A \in \mathrm{Orth}^+$ [$\in \mathrm{Orth}$] and $\tau = 1$, where

$$\text{(3.8)} \qquad \mathrm{Orth} = \{\mathbf{A} \in \mathrm{Lin}_* \mid \mathbf{A}\mathbf{A}^* = \mathbf{1}\}, \qquad \mathrm{Lin}_*^+ = \{\mathbf{A} \in \mathrm{Lin}_* \mid \det \mathbf{A} > 0\},$$
$$\mathrm{Orth}^+ = \{\mathbf{A} \in \mathrm{Lin}_*^+ \mid \mathbf{A}\mathbf{A}^* = \mathbf{1}\}. \text{ (8)}$$

Th. 3.3. *Assume that* $\varphi \equiv (x_\alpha) \in$ CGIIF *and* $\psi \equiv (z_\alpha) \in$ CAIF; *then*

(α) $\Psi \in$ CGIIF *if and only if the trantformation* $\psi \circ \varphi^{-1}$ *is of the type*

$$\text{(3.9)} \qquad \boldsymbol{z} = e\,\delta\,\mathbf{Q}\,\mathbf{x} - \mathbf{b}\,x_0 - \mathbf{c}\,, \qquad z_0 = \tau\,x_0 - c_0 \qquad (e = \pm 1)$$

for some τ, $\delta > 0$ *and* $(\mathbf{Q}, \boldsymbol{b}, \boldsymbol{c}, c_0) \in \mathrm{Orth}^+ \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}$;

(β) *in the case above it results* $h_\psi = \tau^{-2}\,\delta^3\,h_\varphi$;

(γ) *for* $\tau \neq 1$ *the frame* $\varphi^{1\,0\,0}_{\ \ \tau\,0}$ *is not gravitationally equivalent to* φ;

(δ) φ *and* ψ ($\in$ CGIIF) *are gravitationally equivalent if and only if* $\psi \circ \varphi^{-1}$ *is a* (*possibly improper*) *Galilean transformation.*

Proof. Assume that (i) $\varphi \equiv (x_\alpha) \in$ CGIIF, $\psi \equiv (z_\alpha) \in$ CAIF and (ii) $\psi \circ \varphi^{-1}$ is of the type (3.9); choose $t \in \mathbf{R}$, $(\boldsymbol{c}_1, \boldsymbol{c}_2, r_1, r_2) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}_*^+ \times \mathbf{R}_*^+$ with $|\boldsymbol{c}_1 - \boldsymbol{c}_2| > r_1 + r_2$, and let $\mathscr{B} = \mathbf{B}_{\mathscr{P}_1 \curlyvee \mathscr{P}_2}$ and $\mathscr{M}$ be respectively a body and *a* motion for which all conditions in Ax. 3.4 hold. If $\{\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}, t)\}_{(\boldsymbol{y},t) \in \curlyvee^*(\mathscr{P}) \times \mathbf{R}}$ is the φ-representation of $\mathscr{M}$, then

$$\text{(3.10)} \qquad \{\boldsymbol{z} = e\,\delta\,\mathbf{Q}\,\boldsymbol{f}(\boldsymbol{y}, \tau^{-1}(z_0 + c_0)) - \boldsymbol{b}\,\tau^{-1}(z_0 + c_0) - \boldsymbol{c}\} \ldots$$

is its ψ-representation. Hence the acceleration fields in φ and ψ ($\neq \mathbf{0}$ by Lemma 3.2) of the motion $\mathscr{M}$ are related by

$$\text{(3.11)} \qquad [\boldsymbol{a}(\boldsymbol{y}, z_0)]_\psi = e\,\delta\,\tau^{-2}\,\mathbf{Q}\,[\boldsymbol{a}(\boldsymbol{y}, t)]_\varphi;$$

By Def. 3.2 (α) and (3.10-11) – see (α) below (3.4) –, it results

$$\text{(3.12)} \qquad m_1\,[\boldsymbol{a}_1(z_0)]_\psi = e\,\delta\,\tau^{-2}\,\mathbf{Q}\,m_1\,[\boldsymbol{a}_1(t)]_\varphi =$$
$$= e\,\delta\,\tau^{-2}\,\mathbf{Q}\,h_\varphi \int_{\mathscr{P}_{1,t}} \int_{\mathscr{P}_{2,t}} (\boldsymbol{x}_2 - \boldsymbol{x}_1)/|\boldsymbol{x}_2 - \boldsymbol{x}_1|^3\,\mathrm{dm}_2\,\mathrm{dm}_1 =$$
$$= \tau^{-2}\,\delta^3\,h_\varphi \int_{\mathscr{P}_{1,t}} \int_{\mathscr{P}_{2,t}} e\,\delta\,\mathbf{Q}\,(\boldsymbol{x}_2 - \boldsymbol{x}_1)/|\,e\,\delta\,\mathbf{Q}\,(\boldsymbol{x}_2 - \boldsymbol{x}_1)|^3\,\mathrm{dm}_2\,\mathrm{dm}_1 =$$

(8) $\mathbf{1}$ is the identity matrix.

$$= \tau^{-2}\, \delta^3\, h_\varphi \int_{\mathscr{P}_{1,z_0}} \int_{\mathscr{P}_{2,z_0}} (\boldsymbol{z}_2 - \boldsymbol{z}_1)/|\, \boldsymbol{z}_2 - \boldsymbol{z}_1\,|^3 \, \mathrm{dm}_2\, \mathrm{dm}_1 \,,$$

where (α) is assumed and $(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathscr{P}_{1,t} \times \mathscr{P}_{2,t}$, $(\boldsymbol{z}_1, \boldsymbol{z}_2) \in \mathscr{P}_{1,z_0} \times \mathscr{P}_{2,z_0}$. Observe that by Lemma 3.2 $[\boldsymbol{a}_1(z_0)]_\psi \neq \boldsymbol{0} \neq [\boldsymbol{a}_1(t)]_\varphi$. Hence (ψ, h_ψ) with $h_\psi = \tau^{-2}\, \delta^3\, h_\varphi$ satisfies condition (α) in Def. 2.3. It is easily seen that (ψ, h_ψ) satisfies also condition (β) in Def. 2.3. Hence $\psi \in$ CGIIF, and a part of thesis (α) and thesis (β) are proved.

Conversely, assume that $\varphi \equiv (x_\alpha)$ and $\psi \equiv (z_\alpha)$ are in CGIIF; then by Th. 5.3 (α) in [3], $\psi \circ \varphi^{-1}$ has the form $\mathscr{A}\,\mathscr{G}$ where $\mathscr{G}$ is a Galilean transformation, possible involving a change of units, and $\mathscr{A}$ is a spatial affinity (for space-time); hence $\psi \circ \varphi^{-1}$ is of the type

$$\text{(3.13)} \qquad \boldsymbol{z} = \mathbf{AQ}\, \boldsymbol{x} - \boldsymbol{b}\, x_0 - \boldsymbol{c}\,, \qquad z_0 = \tau\, x_0 - c_0\,,$$

for some

$$(\mathbf{A}, \mathbf{Q}, \boldsymbol{b}, \boldsymbol{c}, \tau, c_0) \in \mathrm{Lin}_* \times \mathrm{Orth}^+ \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^+_* \times \mathbf{R}\,.$$

Let $e\,\mathbf{RU}$ be the polar decomposition of $\mathbf{A}$, with $\mathbf{U}$ positive symmetric, $\mathbf{R} \in \mathrm{Ort}^+$, $e = \pm 1$; chose $\hat{\mathbf{R}} \in \mathrm{Orth}^+$ such that $\hat{\mathbf{R}}\, \mathbf{U}\, \hat{\mathbf{R}}^* = \mathbf{D}$ is diagonal with $\mathrm{D}_{rr} = \delta_r > 0$ $(r = 1, 2, 3)$, and set

$$\text{(3,14)} \qquad \chi \equiv (\xi_\alpha)\,, \quad \boldsymbol{\xi} = e\, \mathbf{Q}_0\, \boldsymbol{x} - \mathbf{D}^{-1}\, \mathbf{R}_0^*\, \boldsymbol{b}\, x_0 - \mathbf{D}^{-1}\, \mathbf{R}_0^*\, \boldsymbol{c}\,, \quad \xi_0 = x_0 \quad,$$
$$\text{with } \mathbf{R}_0 = \mathbf{R}\, \hat{\mathbf{R}}^*\,, \ \mathbf{Q}_0 = \hat{\mathbf{R}}\, \mathbf{Q}\,.$$

By the part of thesis (α) already proved, $\chi \in$ CGIIF (as $\varphi \in$ CGIIF) and by (β)

$$\text{(3.15)} \qquad h_\chi = h_\varphi\,;$$

Moreover set

$$\text{(3.16)} \qquad \bar{\psi} = (\bar{z}_\alpha)\,, \quad \bar{\boldsymbol{z}} = \mathbf{R}_0^{-1}\, \boldsymbol{z} \quad, \quad \bar{z}_0 = z_0 \qquad (\mathbf{R}_0 = \mathbf{R}\, \hat{\mathbf{R}}^*)\,.$$

By (3.13–4), as $\mathbf{A} = e\,\mathbf{RU}$ and $\mathbf{D} = \hat{\mathbf{R}}\mathbf{U}\hat{\mathbf{R}}^*$, one deduces that $\bar{\psi} \circ \chi^{-1}$ is

$$\text{(3.17)} \qquad \bar{z} = \mathbf{D}\, \xi\,, \qquad \bar{z}_0 = \tau\, \xi_0 - c_0 = z_0\,, \qquad (\xi_0 = x_0 = t);$$

hence to complete the proof, it suffices to show that

(α) if $\bar{\psi}, \chi \in$ CGIIF and $\bar{\psi} \circ \chi^{-1}$ is of the type (3.17) with $\mathbf{D}$ diagonal, $\mathrm{D}_{rr} = \delta_r > 0$ $(r = 1, 2, 3)$, $\tau > 0$ and $c_0 \in \mathrm{R}$, then $\delta_r = \delta$ for some δ $(r = 1, 2, 3)$.

To this end, choose $t \in \mathbf{R}$, $n \in \mathbf{N}_*$ and $(\boldsymbol{c}_1, \boldsymbol{c}_2) \in \mathbf{R}^3 \times \mathbf{R}^3$ with $|\,\boldsymbol{c}_1 - \boldsymbol{c}_2\,| > 2$. By Ax. 3.4 there exist $\mathscr{P}_1^n, \mathscr{P}_2^n \in \mathscr{M}\mathscr{P}$ and a motion $\mathscr{M}^n$ for the body $\mathscr{B}^n =$

$= \mathbf{B}_{\mathscr{P}_1^n \curlyvee \mathscr{P}_2^n}$ which, at instant t, satisfy conditions (i) to (iii) in Ax. 3.4 with $r_1 = r_2 = 1/n$. Let $\{\xi = f_n(y, \xi_0)\}_{(y,\xi_0) \in \curlyvee^*(\mathscr{P}_1^n \curlyvee \mathscr{P}_2^n) \times \mathbf{R}}$ be the χ-representation of $\mathscr{M}^n$; then its $\bar{\psi}$-representation is $\{z = \mathbf{D} f_n(y, \tau^{-1}(z_0 + c_0))\} \ldots$ – see (3.17) –, and the acceleration field has $\bar{\psi}$ and χ – representations related by

$$[a(y, z_0)]_{\bar{\psi}} = \tau^{-2} \mathbf{D} [a(y, \xi_0)]_\chi \qquad (\xi_0 = t). \tag{3.18}$$

By (3.4), (3.17-18) and as $h_{\bar{\psi}} = h_\psi$, it results

$$\begin{aligned} \tau^{-2} \mathbf{D}\, \mathrm{m}_1 [a_1(y, t)]_\chi &= [a_1(y, z_0)]_{\bar{\psi}} = \\ &= h_\psi \int_{\mathscr{P}^n_{1,z_0}} \int_{\mathscr{P}^n_{2,z_0}} (z_2 - z_1)/|\, z_2 - z_1 |^3 \,\mathrm{dm}_2\, \mathrm{dm}_1 = \\ &= h_\psi \int_{\mathscr{P}^n_{1,t}} \int_{\mathscr{P}^n_{2,t}} \mathbf{D}(\xi_2 - \xi_1)/|\, \mathbf{D}(\xi_2 - \xi_1)|^3 \,\mathrm{dm}_2\, \mathrm{dm}_1 . \end{aligned} \tag{3.19}$$

Again by (α) in Def. 3.2.

$$\tau^{-2} \mathbf{D}\, m_1 [a_1(y, t)]_\chi = \tau^{-2} \mathbf{D}\, h_\chi \int_{\mathscr{P}^n_{1,t}} \int_{\mathscr{P}^n_{2,t}} (\xi_2 - \xi_1)/|\, \xi_2 - \xi_1 |^3 \,\mathrm{dm}_2\, \mathrm{dm}_1 ,$$

hence by $(3.19)_{1,3}$

$$\begin{aligned} \tau^{-2} h_\chi \int_{\mathscr{P}^n_{1,t}} \int_{\mathscr{P}^n_{2,t}} (\xi_2 - \xi_1)/|\, \xi_2 - \xi_1 |^3 \,\mathrm{dm}_2\, \mathrm{dm}_1 &= \\ = h_\psi \int_{\mathscr{P}^n_{1,t}} \int_{\mathscr{P}^n_{2,t}} (\xi_2 - \xi_1)/|\, \mathbf{D}(\xi_2 - \xi_1)|^3 \,\mathrm{dm}_2\, \mathrm{dm}_1 . \end{aligned}$$

Divide this equality by $m_1 m_2$, and take its limit for $n \to \infty$; by (3.15), one obtains

$$\tau^{-2} h_\varphi (c_2 - c_1)/|\, c_2 - c_1 |^3 = h_\psi (c_2 - c_1)/|\, \mathbf{D}(c_2 - c_1)\, |^3; \tag{3.20}$$

we conclude that this equality holds for all c_1, c_2 with $|\, c_1 - c_2 \,| > 2$. For $r = 1, 2, 3$ choose $c_2 - c_1 = (c_2 - c_1)\, i_r$; then by (3.20) it follows that $\tau^{-2} h_\varphi = h_\psi\, \delta_r^{-3}$; hence $\delta_r = \delta$ $(r = 1, 2, 3)$ for some δ. Thus (α) is completely proved [9]. To prove (γ) observe only that if $\psi = \varphi^{1}{}_{\tau}{}^{0}{}_{0}{}^{0}$ – see (3.7) – is gravitationally equivalent to φ then by the same reasonings above, (3.20) must hold

(9) By setting in (3.13) $\mathbf{D} = \delta\, \mathbf{1}$, $\mathbf{A} = e\, \mathbf{R}\, \mathbf{U}$, $\mathbf{U} = \hat{\mathbf{R}}^* \mathbf{D}\, \hat{\mathbf{R}}$, $\mathbf{R}_0 = \mathbf{R}\, \hat{\mathbf{R}}^*$ and $\mathbf{Q}_0 = \hat{\mathbf{R}}\, \mathbf{Q}$, one obtains $z = e\, \delta\, \mathbf{R}_0\, \mathbf{Q}_0\, x - b x_0 - c$.

with $h_\varphi = h_\psi$ (see Th. 3.3 (α), (β)) and $\mathbf{D} = \mathbf{1}$, which implies $\tau = 1$. Lastly (δ) is a corollary of (α)-(γ). q.e.d.

According to [3, N. 5] and [2, p. 173], for $\varphi \in$ CAIF Σ_φ denotes the set of those PWIMPs that have zero φ-velocity – see [3, Def. 3.5] –; thus, exactly as in [2, p. 173], Σ_φ receives the structure of a 3-dimensional affine space and is said to be an *affine inertial space*. Furthermore, for $t \in \mathbf{R}$, φ-inst$_t$ defined as $\{\mathscr{E} \in \mathrm{EP} \mid \varphi_0(\mathscr{E}) = t\}$ (10) is said to be the φ- *instant of absciss* t; via the natural bijection $b : \Sigma_\varphi \to \varphi$-Inst$_t$ for which $b(l) = l \cap \varphi$-Inst$_t$, φ-Inst$_t$ *appears to be an affine space of simultaneous events.*

For (i) $\varphi \equiv (x_\alpha) \in$ CGIIF, (ii) $l, l' \in \Sigma_\varphi$, (iii) $\mathscr{E} \in l$, $\mathscr{E}' \in l'$, $x_i = \varphi_i(\mathscr{E})$, and $x'_i = \varphi_i(\mathscr{E}')$, set

$$\mathrm{d}_\varphi(l, l') = \delta_\varphi(\mathscr{E}, \mathscr{E}') = |\boldsymbol{x} - \boldsymbol{x}'| = [\sum_{i=1}^{3} (x_i - x'_i)^2]^{1/2} . \tag{3.19}$$

On the basis of Th. 3.3 (δ) it results that $|\boldsymbol{x} - \boldsymbol{x}'|$ *is independent of* φ *in the class of (gravitationally) isotropic inertial frames (gravitationally) equivalent to a given* $\varphi_0 \in$ CGIIF; thus (i) $\delta_\varphi(\mathscr{E}, \mathscr{E}')$ is called *spatial* φ*-distance* between the events $\mathscr{E}$ and $\mathscr{E}'$; and (ii) $\mathrm{d}_\varphi(l, l')$ is called *spatial* φ- *distance* between the inertial points l and l'. On the basis of the above considerations one deduces the

Th. 3.4. *Assume that* φ *and* ψ *be in* CGIIF; *then*

(α) l *and* $l' \in \Sigma_\varphi = \Sigma_\psi$ *imply* $\mathrm{d}_\varphi(l, l') = \mathrm{d}_\psi(l, l')$; *and*

(β) *if* $\mathscr{E}, \mathscr{E}' \in \varphi$*-inst*$_t$ *for some* t, *then* $\delta_\varphi(\mathscr{E}, \mathscr{E}') = \delta_\psi(\mathscr{E}, \mathscr{E}')$ *whenever* $\psi \circ \varphi^{-1}$ *is a Galilean transformation* – see below (3.7) –.

Hence one can speak, in particular, of the *physical Euclidean metric* defined on each inertial space [on each φ-instant], and Σ_φ [φ-Inst$_t$] appears to be an *Euclidean inertial space* [an *Euclidean space of simultaneous events*].

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(10) EP is the set of event points, i.e. space-time.