Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Alberto Bressan, Agostino Cortesi

Lipschitz extensions of convex-valued maps

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **80** (1986), n.7-12, p. 530–532.

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<http://www.bdim.eu/item?id=RLINA_1986_8_80_7-12_530_0>

Analisi matematica. — *Lipschitz extensions of convex-valued maps.*
Nota (*) di ALBERTO BRESSAN (**) e AGOSTINO CORTESI (***), presentata dal Corrisp. R. CONTI.

RIASSUNTO. — Si dimostra che ogni funzione multivoca lipschitziana con costante di Lipschitz M, definita su un sottoinsieme di uno spazio di Hilbert H a valori compatti e convessi in $\mathbf{R}^n$, può essere estesa su tutto H ad una funzione multivoca lipschitziana con costante minore di 7 nM. In generale, non esistono invece estensioni aventi la stessa costante di Lipschitz M.

1. INTRODUCTION

A classical theorem by Kirszbraun-Valentine [3, p. 52] states that every Lipschitz continuous map f from a subset A of a Hilbert space H into a Hilbert space H′ has an extension $\tilde{f}: H \to H'$ with the same Lipschitz constant. The present paper is concerned with the analogous extension problem for convex-valued multifunctions.

Let F be a multivalued map defined on a subset A of a Hilbert space H, whose values are compact convex subset of $\mathbf{R}^n$. We prove that if F is Lipschitz continuous with constant M, then it admits a multivalued extension $\tilde{F}: H \to \mathbf{R}^n$ with Lipschitz constant smaller than $7\,n\,M$. The proof combines Kirszbraun's extension theorem with a selection lemma due to Le Donne and Marchi [2].

A counterexample shows that, in general, it is not possible to construct an extension $\tilde{F}$ preserving the same Lipschitz constant M.

2. THE MAIN RESULT

Let H be a Hilbert space and call Ω^n the space of all compact, convex subsets of $\mathbf{R}^n$ provided with the Hausdorff metric d_H. We shall write $\overline{co}\,A$ for the closed convex hull of the set A and $B(K, \varepsilon)$ for the closed ε-neighbourhood around the set $K \subset \mathbf{R}^n$. For the basic properties of multivalued maps we refer to [1]. The following selection lemma was proved in [2].

(*) Pervenuta all'Accademia il 28 ottobre 1986.
(**) Dept. of Mathematics, Un. of Colorado, Boulder, Co. 80309 U.S.A.
(***) Seminario Matematico, Un. di Padova, via Belzoni 7, 35100, Italy.

LEMMA. *Let* $\overline{K} \in \Omega^n$ *and* $\hat{y} \in \overline{K}$. *Then there exists a Lipschitz continuous map* $S_{\overline{K},\hat{y}}$ *from* Ω^n *into* $\mathbf{R}^n$ *with constant* $2\,n\,L$ *such that* $S_{\overline{K},\hat{y}}(K) \in K$ *for all* $K \in \Omega^n$ *and* $S_{\overline{K},\hat{y}}(\overline{K}) = y$.

Here L is a constant determined by the system

$$\begin{cases} \alpha + \beta = \pi/3 \\ \sin\ \alpha = 1/L \\ \sin\ \beta = 2/L \end{cases}$$

with $\alpha < \beta < \pi/2$. A direct computation yields $L = \sqrt{28/3}$.

THEOREM. *For any* $A \subseteq H$, *every Lipschitz continuous map* $F : A \to \Omega^n$ *with constant* M *can be extended to a map* $\tilde{F} : H \to \Omega^n$ *with Lipschitz constant* $2\,n\,M\sqrt{28/3}$.

Proof. For every $\hat{x} \in A$ and $\hat{y} \in F(\hat{x}) \in \Omega^n$, let $S_{F(\hat{x}),\hat{y}} : \Omega^n \to \mathbf{R}^n$ be the Lipschitz selection provided by the previous lemma.

The composition $f_{\hat{x},\hat{y}} = S_{F(\hat{x}),\hat{y}} \circ F$ is then a continuous (single-valued) map from A into $\mathbf{R}^n$ with Lipschitz constant $2n\,ML$, hence it admits an extension $\tilde{f}_{\hat{x},\hat{y}} : H \to \mathbf{R}^n$ with the same Lipschitz constant [3, p. 52].

Define $\tilde{F} : H \to \Omega^n$ by setting

$$\tilde{F}(x) = \overline{co}\,\{\tilde{f}_{\hat{x},\hat{y}}(x)\ ;\ \hat{x} \in A\,,\ \hat{y} \in F(\hat{x})\} \tag{1}$$

Notice that $\tilde{F}(x) = F(x)$ whenever $x \in A$. Indeed, for $x \in A$ we have

$$\tilde{F}(x) \supseteq \{\tilde{f}_{x,y}(x)\ ;\ y \in F(x)\} = \{f_{x,y}(x)\ ;\ y \in F(x)\} = F(x)\,,$$

$$\tilde{F}(x) = \overline{co}\,\{f_{\hat{x},\hat{y}}(x)\ ;\ \hat{x} \in A\,,\ \hat{y} \in F(\hat{x})\} \subseteq F(x)$$

because $f_{\hat{x},\hat{y}}(x) \in F(x)$ for all $\hat{x}, \hat{y}$, and $F(x)$ is closed and convex.

It remains to check the Lipschitz continuity of $\tilde{F}$, i.e.

$$d_H(\tilde{F}(x)\,,\ \tilde{F}(x')) \leq 2n\,M\,L\,\|x - x\|\,,\ \forall\,x\,,\,x' \in H \tag{2}$$

To prove (2) observe that

$$d_H(\tilde{F}(x)\,,\,\tilde{F}(x')) \leq d_H(\{\tilde{f}_{u,v}(x)\ ;\,u \in A\,,\,v \in F(u)\}\,,\,\{\tilde{f}_{u,v}(x')\ ;\,u \in A\,,\,v \in F(u)\})\,. \tag{3}$$

If (2) fails, then by (3) there exist $x\,,\,x' \in H$, $\hat{x} \in A$ and $\hat{y} \in F(\hat{x})$ for which

$$\tilde{f}_{\hat{x},\hat{y}}(x) \notin B(\{\tilde{f}_{u,v}(x')\ ;\ u \in A\,,\ v \in F(u)\}\,,\ 2\,n\,M\,L\,\|x - x'\|)\,.$$

In particular, $\| \tilde{f}_{\hat{x},\hat{y}}(x) - \tilde{f}_{\hat{x},\hat{y}}(x') \| > 2n\,\mathrm{ML}\, \| x - x' \|$, which contradicts the Lipschitz continuity of $\tilde{f}_{\hat{x},\hat{y}}$.

Therefore (2) holds, and the proof is completed.

3. A counterexample

In general, Lipschitz continuous convex-valued maps do not admit extensions which preserve the same Lipschitz constant.

Example. Let P_1, P_2, P_3 be three vertices of an equilateral triangle T in a plane, with side length 1, i.e. $| P_i - P_j | = \delta_{ij}$. Let $Q = (P_1 + P_2 + P_3)/3$ be the barycenter of T, and define

$$F(P_1) = \{ (x, y) \in \mathbf{R}^2 ; x = 0 , 0 \leq y \leq 1 \} .$$

$$F(P_2) = \{ (x, y) \in \mathbf{R}^2 ; 0 \leq x \leq 1 , y = 0 \} ,$$

$$F(P_3) = \{ (x, y) \in \mathbf{R}^2 ; 0 \leq x \leq 1 , 0 \leq y \leq 1 \} .$$

Clearly, $d_H(F(P_i), F(P_j)) = \delta_{ij}$, hence the map F is non-expansive on the set $\{P_1, P_2, P_3\}$. Assume that some set F(Q) could be defined in such a way that

$$d_H(F(Q), F(P_i)) \leq | Q - P_i | = 1/\sqrt{3}, \qquad i = 1, 2, 3 . \tag{4}$$

We claim that this leads to a contradiction. Indeed, using (4) with $i = 1, 2$, we get

$$F(Q) \subseteq [B(F(P_1), 1/\sqrt{3}) \cap B(F(P_2), 1/\sqrt{3})] = \Gamma$$
$$= \{ (x, y) ; x^2 + y^2 \leq 1/3 \} \cup \{ (x, y) ; 0 \leq x \leq 1/\sqrt{3} , 0 \leq y \leq 1/\sqrt{3} \} .$$

Since $(1,1) \in F(P_3)$, one has

$$d_H(F(P_3), F(Q)) \geq d((1, 1), F(Q)) \geq d((1, 1), \Gamma) =$$
$$= (1 - 1/\sqrt{3})\sqrt{2} > 1/\sqrt{3} ,$$

which is in contradiction with (4), $i = 3$.

References

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