Consequences of compactness properties for abstract logics


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RIASSUNTO. — Si determinano alcune restrizioni sulle possibili cardinalità dei modelli di teorie in logiche soddisfacenti alcune proprietà di compattezza. Si dà una caratterizzazione delle logiche $[\lambda, \mu]$-compatte generate da quantificatori di cardinalità. Si stabilisce che il primo cardinale $k$ tale che una logica $\mathcal{L}$ è $(k, k)$-comatta è debolmente inaccessibile e soddisfa la proprietà dell’albero. Dai risultati enunciati appare un confronto assai particolareggiato fra i due concetti di $(\lambda, \mu)$-compattezza e $[\lambda, \mu]$-compattezza.

For an Abstract Logic $L$, [4] introduced the notion of $[\lambda, \mu]$-compactness which is, in many respects, more natural than $(\lambda, \mu)$-compactness (nevertheless, when one is interested in logics which are not fully compact, this new compactness property seems much stronger than the older one: as an example, $L_{\omega_0} (Q_1)$ is $(\omega, \omega)$ but not $[\omega, \omega]$-compact).

The aim of this paper is to announce some results about $[\lambda, \mu]$-compactness and, in most cases, alternative forms for $(\lambda, \mu)$-compactness are given. At least, we show that the new notion is not only interesting in itself, but can also be used to suggest new theorems about $(\lambda, \mu)$-compactness.

Unexplained notions and notations can be found in [1], [2], [4]; $\lambda$, $\mu$, $\nu$, $k$ denotes infinite cardinals; if $L$ is a logic, $F_\nu (L)$ is the class of all couples $(D, V)$, $D$ an ultrafilter and $V$ filter such that $\prod_{D|V} \mathcal{L} = L$ $\forall D|V$ with $\forall D|V$ $\exists \exists (D, V) \in F_\nu (L)$, $(D, V)$ $(\lambda, \mu)$-regular, such that $\inf \{K \cap [\mu^*, \omega]\} \geq |\prod_{D|V} \mu'| = |\prod_{D|V} \mu| > \mu^*$, for every $\mu' \in [\mu, \mu^*]$.

(*) Nella seduta del 29 novembre 1986.
COROLLARY 1. There is no logic generated by monadic and equivalence quantifiers satisfying the Relativized Upward Lowenheim Skolem Property and properly extending $L_{\omega_{2\omega}}$.

Corollary 1 strengthens [1, Ch. VI, Theorem 3.1.3] (but compare also with [3, p. 236]). The method of proof of Corollary 1 can be applied to many other kinds of quantifiers.

In view of Theorem 1, a $[\lambda, \omega]$-compact logic $L$ either is rich in $L$-complete extensions of small cardinality, or satisfies many Lowenheim-Skolem-Tarsky properties. A counterpart for $(\lambda, \omega)$-compactness is:

THEOREM 2. Suppose that $L$ is $(\lambda, \omega)$-compact, and put $K = \{k \mid$ for every $L$-theory $T$ with $|T| \leq \lambda$, and for every unary predicate $U$, if every finite subset of $T$ has a model in which $|U| \leq k$, then $T$ has a model in which $|U| \leq k\}$. Then, if $|T| \leq \lambda$, $(U_{a})_{a \in \lambda}$ are unary predicates, $[\mu, \mu'] \cap K = \emptyset$, and every finite subset of $T$ has a model in which $|U_{a}| \in [\mu, \mu'] (a \in \lambda)$, then $T$ has a model in which $\inf (K \cap (\mu', \infty)) \geq |U_{a}| => |U_{b}| > \mu'$, for every $a, b \in \lambda$.

THEOREM 3. Suppose that $L$ is single-sorted and $(\lambda, \omega)$-compact, and satisfies the Craig Interpolation Property. Then either: (i) $L$ contains a sentence of empty type not in $L_{\omega_{2\omega}}$ or (ii) if $(T_{a})_{a \in \lambda}$ are $L$-theories, $|T_{a}| \leq \lambda (a \in \lambda)$, each having an infinite model, then they have models of the same infinite power.

In Theorem 3 we do not need the hypothesis that $L$ is closed under relativization, which is essential in all the other theorems.

If $N$ is a logic, let $N_{\lambda\mu}$ be the logic obtained by $N$, admitting conjunctions and disjunctions of less than $\lambda$ sentences, and quantifications over less than $\mu$ constants. The following generalizes a result of [4].

THEOREM 4. If $\mu$ is the first cardinal such that $N$ is $[\mu, \mu]$-compact, then also $N_{\mu\mu}$ is $[\mu, \mu]$-compact, and $\mu$ is a measurable cardinal.

Theorem 4 cannot be extended to $N_{\mu\mu}$: if $\mu$ is an uncountable measurable cardinal, $L_{\mu\mu} (Q^{\mu\mu})$ is $[\mu, \mu]$-compact, but $L_{\mu\mu} (Q^{\mu\mu})$ is not $[\mu, \mu]$-compact. Nevertheless, we have an analogue for $(k, k)$-compactness:

THEOREM 5. If $k$ is the first cardinal such that $N$ is $(k, k)$-compact, then $N_{k\mu}$ is still $(k, k)$-compact, so that $k$ is weakly inaccessible and has the tree property. If, in addition, $k$ is strong limit, then $k$ is weakly compact.

THEOREM 6. If $K$ is any class of cardinals, then $L_{\omega_{0}} (Q_{\alpha})_{\alpha \in K}$ is $[\lambda, \mu]$-compact iff there exists a $(\lambda, \mu)$-regular not (cf $\omega_{\alpha}$, cf $\omega_{\beta}$)-regular $(\alpha \in K)$ ultrafilter $D$ such that $k < \omega_{\alpha}$ implies $\prod_{D} k < \omega_{\alpha} (\alpha \in K)$.
Theorem 6 shows an influence of set-theoretical axioms (concerning the existence of non-regular ultrafilter) on the problem of compactness of cardinality logics. Set theory influences also the possible compactness spectrum of logics:

**Theorem 7.** If $I$, $J$ are sets, and the $\nu_j$'s are regular cardinals, then the following are equivalent:

(i) There exists a logic $[\lambda_i, \mu_i]$-compact ($i \in I$) not $[\nu_j, k_j]$-compact ($j \in J$).

(ii) There exists a logic as in (i) generated by a set of cardinality quantifiers.

(iii) For every $j \in J$ there exists $\nu_j^*, k_j \leq \nu_j^* \leq \nu_j$, such that for every $i \in I$ there is an ultrafilter which is $(\lambda_i, \mu_i)$-regular but not $(\nu_j^*, \nu_j^*)$-regular, for $j \in J$.

Theorem 7 improves [3, Lemma 6.4 (ii)].

In many particular cases, Theorem 6 can be used in order to give a more explicit characterization of $[\lambda, \mu]$-compact cardinality logics. An example is:

**Theorem 8.** If $k$ is strongly compact (or just $\sup (k, \omega_a)$-compact) and $\lambda \geq k$ is regular, then $L_{\omega_\omega} (Q_a)$ is $[\lambda, k]$-compact iff $\text{cf} (\omega_a) \notin [k, \lambda]$ and $\nu < \omega_a$, for all $\nu < \omega_a$ with $\text{cf} \nu > k$.

**Bibliography**


