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Collection of charged grains by gravitational and electromagnetic action of a planet

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Astronomia. — *Collection of charged grains by gravitational and electromagnetic action of a planet.* Nota di VITTORIO BANFI, presentata (*) dal Socio M.G. FRACASTORO.

RIASSUNTO. — È presentato uno studio che riguarda la possibile raccolta di piccoli granelli materiali, con dimensione significativa dell'ordine del micron e dotati di carica elettrica, da parte di un pianeta sede di campo magnetico. Si studia un modello che tiene conto dell'azione combinata, gravitazionale ed elettromagnetica, responsabile di tale effetto.

Le orbite stabili di questi granelli, provenienti dallo spazio interplanetario, si dispongono nel piano equatoriale del pianeta e sono di forma circolare. Come è intuitivo, nella prima parte del suo avvicinamento al pianeta la particella è soggetta, in modo preponderante, alla forza gravitazionale, mentre nell'ultima parte, al contrario, alla forza elettromagnetica.

Il risultato finale è un disco di materiale disaggregato e neutro, prevalentemente disposto nel piano equatoriale; i satelliti dovrebbero formarsi a partire da questo disco, in base a meccanismi di accrezione studiati da vari autori.

È interessante notare che è dimostrato essere l'estensione del disco una funzione a due variabili: la carica elettrica della particella e la sua massa, una volta che siano fissati il campo magnetico e la massa del pianeta.

Lo studio del moto solo nel piano equatoriale non costituisce una limitazione del problema, essendo lo scopo della ricerca di tipo cosmogonico Infatti il materiale neutro raccolto non può disporsi, restando in equilibrio dinamico, altro che nel piano equatoriale.

1. INTRODUCTION

Kinetics effects, caused by a combination of the gravitational force and of the electromagnetic force on micron-sized dust particles, have been extensively studied recently by several authors (Mendis *et al.*, 1982, Northrop *et al.*, 1983).

The purpose of these researches is to describe carefully the motion of the grains in the ring systems and in the rapidly rotating magnetosphere of the outer planets, such as Jupiter and Saturn. Interactions between charged grains and surrounding plasma are also studied.

From an analytical point of view the equation of motion of the grain is written in a frame of reference fixed and centred in the planet. It was supposed

(*) Nella seduta del 20 giugno 1986.

that the grain, at the beginning, is within the region of the magnetosphere, where the plasma is rotating.

In the present work we have a different aim and, of course, a different analytical model. The goal of this investigation is to demonstrate the possible accretion of the micron-sized charged grains by any planet (having magnetic field), caused by an interplay of gravitational and electrodynamical effects. In other words, the protoplanet captures some dusty material (incoming from the interplanetary space) in order to build up a satellite; therefore, the present investigation has a cosmogonical nature. Also the analytical approach is different. In fact, it is of a point-dynamical type and the frame of reference is an absolute one, having the point-mass planet at the origin. Obviously, there are common statements deriving from two types of approach; the fundamental one is that circular orbits are possible in the planet's equatorial plane.

2. General

Consider a central massive body (like Jupiter, for instance) emanating a magnetic field: it is possible to show that small solid particles (with electric charge) are gathered by the central body, at some fixed distance, in the equatorial plane. Consequently, after recombination of this kind of " dusty plasma ", the particles, having lost their electric charges, will arrange themselves on a disc lying on the equatorial plane of the planet.

In order to demonstrate this, we must make the following assumptions:

— the small solid particles have a spherical shape, with an electric charge uniformly distributed throughout their surface;

— the radius of the particles is in the range 1-5 μm and their density is 2×10^3 Kg/m (they have been detected by spacecraft);

— the motion of the particles can be studied according to the standard mass-point dynamic theory;

— Poynting-Robertson effects are neglected, because the central body is a planet.

The presence of electric charge on the particles is due to the solar wind effects or, more generally, to cosmic plasmas, within interplanetary spaces, in the primeval epoch of Solar System formation (Alfvén, 1975). The single charge of the particles was assumed equal to 10^{-15} coulombs.

3. Analytical approach to the problem

The charged solid particle is submitted to the planet gravitation and to the Lorentz force due to magnetic field. Last term, the vector $\overline{B}$, is provided by the formula

$$\overline{B} = \frac{a}{r^3}\,\overline{n}\,, \tag{1}$$

where $\overline{n}$ is the unit vector, located perpendicularly to the equatorial plane (plane π in fig. 1). In this plane we consider the particle motion; r is its distance from origin 0. Since the MKS unit system is used, the term a is equal to $\mu_0 \dfrac{M}{2\pi}$, M being the true magnetic moment of the dipole (Kraus, 1953).

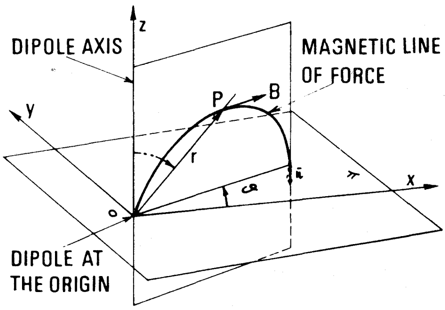

Fig. 1. – Geometry of the magnetic field emanating from the planet, whose centre is in origin 0.

The vectorial equation of motion is then:

$$m\,\frac{d\overline{v}}{dt} = -\,G\,\frac{M_p\,m}{r^2}\,\hat{r} + q\overline{v} \times \overline{B}\,, \tag{2}$$

where:

m = mass of the particle,
M_p = mass of the planet,
$\overline{v}$ = vector velocity of the particle,
$\hat{r}$ = unit vector of the direction r,
q = electric charge of the particle,
G = gravitational constant.

Assuming r, θ as polar coordinates in the π plane (fig. 1), we shall have two scalar equations instead of (2):

$$\ddot{r} = -\,\frac{GM_p}{r^2} - a\,\frac{q}{m}\,\frac{\dot{\theta}}{r^2} + r\dot{\theta}^2 \tag{3}$$

$$\frac{1}{r}\,\frac{d}{dt}\left(r^2\,\dot{\theta}\right) = \frac{q}{m}\,a\,\frac{\dot{r}}{r^3}\,. \tag{4}$$

Eq. (4) is readily integrated

$$r^2 \dot\theta = + \int_\infty^r \frac{q}{m} \frac{a}{r^2} \, \mathrm{d}r = - \frac{q}{m} \frac{a}{r} \tag{5}$$

Initial conditions of Eq. (5) were: at $t = 0$, $r \to \infty$ $\left(\text{or, more practically, } r = r_A \text{ very large in order to neglect the term } \frac{qa}{mr_A}\right)$; at $t = 0$, $r^2 \dot\theta$ (angular momentum for mass unit) equal to zero. Putting (5) into (3) we obtain

$$\ddot r = - \frac{\mathrm{GM}_p}{r^2} + 2 \frac{q^2}{m^2} \frac{a^2}{r^5} \tag{6}$$

Now we can write:

$$\ddot r = \frac{\mathrm{d}\dot r}{\mathrm{d}t} = \frac{\mathrm{d}\dot r}{\mathrm{d}r} \frac{\mathrm{d}r}{\mathrm{d}t} = \dot r \frac{\mathrm{d}\dot r}{\mathrm{d}r},$$

and inserting it into Eq. (6), we have

$$\dot r \frac{\mathrm{d}\dot r}{\mathrm{d}r} = - \frac{\mathrm{GM}_p}{r^2} + 2 \frac{q^2}{m^2} \frac{a^2}{r^5}.$$

The integration of the preceding equation gives:

$$\dot r^2 = 2 \frac{\mathrm{GM}_p}{r} - \frac{q^2}{m^2} \frac{a^2}{r^4} \tag{7}$$

assuming $\dot r = 0$, at $t = 0$ and $r = r_A \cong \infty$. From Eq. (7) we note that for

$$r = r_m = \sqrt[3]{\frac{a^2 q^2}{m^2} \frac{1}{2\mathrm{GM}_p}} \neq 0 \tag{8}$$

it is $\dot r = 0$.

Going back to Eq. (7), for $k_0 = 2\mathrm{GM}_p$ and $k_1 = \frac{a^2 q^2}{m^2}$, we have:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = - \sqrt{\frac{k_0 r^3 - k_1}{r^4}} < 0 \tag{9}$$

(because the particle quite certainly will fall towards the planet). Integrating between 0 and t, r_A and r, we obtain:

$$\frac{1}{3 k_0} \int_{r_A}^{r} \frac{3 r^2 k_0}{\sqrt{k_0 r^3 - k_1}} \, \mathrm{d}r = - t$$

and then

$$\sqrt{k_0 r_A^3 - k_1} - \sqrt{k_0 r^3 - k_1} = \frac{3}{2} k_0 t . \tag{10}$$

Putting $A_0 = \sqrt{k_0 r_A^3 - k_1}$ Eq. (10) gives:

$$A_0 - \frac{3}{2} k_0 t = \sqrt{k_0 r^3 - k_1} . \tag{11}$$

Squaring the preceding equation and rearranging the various terms, we obtain

$$r = \sqrt[3]{r_A^3 - 3A_0 t + \frac{9}{4} k_0 t^2} \tag{12}$$

which provides the law of variation of $r(t)$. Searching the value of t which makes $\dot{r} = 0$ we have

$$\frac{d}{dt}\left(r_A^3 - 3A_0 t + \frac{9}{4} k_0 t^2\right) = 0$$

and then

$$t_m = \frac{2A_0}{3k_0} . \tag{13}$$

Inserting t_m into (12) we have obviously (8). It is useful to write Eq. (12) in normalized form; then we obtain:

$$\eta = \frac{r}{r_A} = \sqrt[3]{1 + \alpha_1 x + \alpha_2 x^2} \tag{14}$$

here

$$\alpha_1 = - \frac{2A_0^2}{k_0 r_A^3} \quad , \quad \alpha_2 = \frac{A_0^2}{k_0 r_A^3} \quad , \quad x = \frac{t}{t_m} .$$

Now we must integrate Eq. (5). From Eq. (11) we obtain:

$$r^3 = \frac{1}{k_0}\left[\left(A_0 - \frac{3}{2} k_0 t\right)^2 + k_1\right].$$

Therefore, Eq. (5) becomes

$$\frac{d\theta}{dt} = \frac{k_0 \sqrt{k_1}}{\left(A_0 - \frac{3}{2} k_0 t\right)^2 + k_1}, \tag{15}$$

which we must integrate with the following initial conditions: at $t = 0$, $\theta = 0$. Therefore, we have

$$\theta = k_0 \sqrt{k_1} \int_0^t \frac{d\,t}{\left(A_0 - \frac{3}{2} k_0 t\right)^2 + k_1}$$

and performing the integration we shall obtain:

$$\theta = \frac{2}{3} \arctan \frac{\frac{A_0}{\sqrt{k_1}} \frac{t}{t_m}}{1 - \frac{A_0^2}{k_1}\left(\frac{t}{t_m} - 1\right)}. \tag{16}$$

Inserting $x = t/t_m$ and $\alpha = A_0/\sqrt{k_1}$, we obtain Eq. (16) in normalized form, namely

$$\theta = \arctan \frac{\alpha x}{1 - \alpha^2 (x - 1)}. \tag{17}$$

Summarizing the foregoing analytical developments, we can say that the motion is thus known by means of Eqs. (14) and (17). In fact, the particle will fall, towards the planet between $t = 0$ and $t = t_m$. Starting from the instant $t = t_m$ it will circulate, with constant tangential velocity

$$v_T = a \frac{q}{m} \frac{1}{r_m^2}. \tag{18}$$

This velocity is the well known " Larmor velocity "; in fact, from Eq. (18) we have:

$$v_T = a \frac{q}{m} \frac{r_m}{r_m^3} = B_m r_m \frac{q}{m},$$

i.e. the circular uniform motion of the particle within a uniform magnetic field of induction B_m (the quantity v_T/r_m is also commonly called the " cyclotron frequency ").

4. Some interpretations and numerical examples

Let us consider the planet Jupiter. From the measurements made by Voyager Spacecraft, the magnetic induction, at a distance 4.2×10^8 m from Jupiter centre, can be assumed to be $B = 30 \times 10^{-5}$ tesla. Then it follows $a = 2.22 \times 10^{22}$ tesla m^3. A particle, with radius $r_0 = 2 \times 10^{-6}$ m and density $\rho = 2 \times 10^3$ Kg m^{-3}, has a mas $m = 0.67 \times 10^{-13}$ Kg. Then we have $k_1 = 11 \times 10^{42}\, m^6 s^{-2}$, $k_0 = 255 \times 10^{15}$ m$^3 s^{-2}$. From these figures, it follows $r_m = 3.5 \times 10^5$ Km. This value appears an acceptable one being of the same order of magnitude 4×10^8 m (the orbit radius of satellite Io). Furthermore,

$$A_0 = 3.4 \times 10^{25} \text{ m}^3 s^{-1} \qquad \alpha = 1.03 \times 10^4$$

$$t_m = 2.85 \text{ yrs} \qquad \alpha_1 = -2 \qquad \alpha_2 = 1 .$$

After computing these figures, it is very easy to plot Eqs. (14) and (17). These graphs are shown in figs. 2 and 3. Note the very abrupt transition (fig. 3), something like an ideal step between a value close to zero (till $x > 0.9$) and a value of 90° (final value).

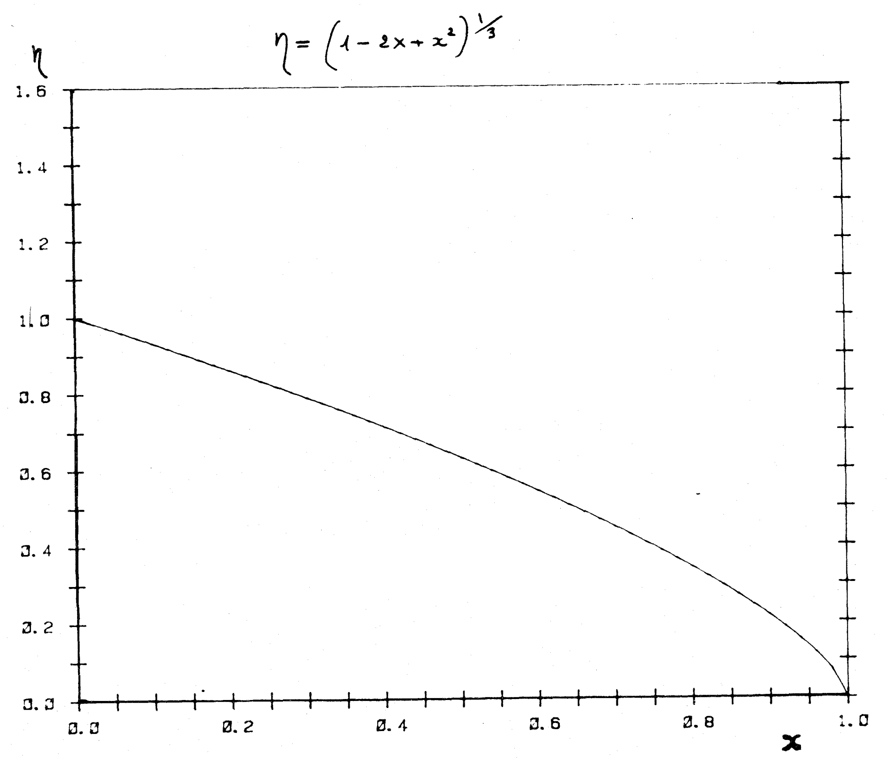

Fig. 2. – Plot of the function $\eta = \eta(x)$, provided by Eq. [14]. The value of η, for $x = 1$, is negligible in comparison with 1 (since $r_m \ll r_A$).

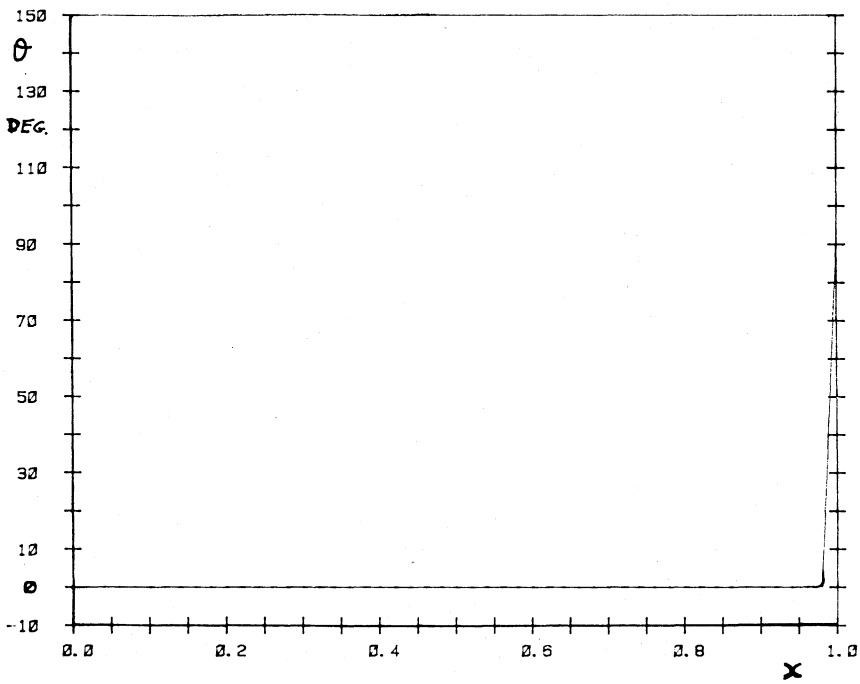

Fig. 3. – Plot of the function $\vartheta = \vartheta(x)$, provided by Eq. [17].

From a qualitatively point of view, we can summarize the main features of the motion in the plan r, θ (fig. 4). For a positive electrically-charged particle the trajectory (1) is composed of two parts: (a) between $r = r_A$ and $r =$

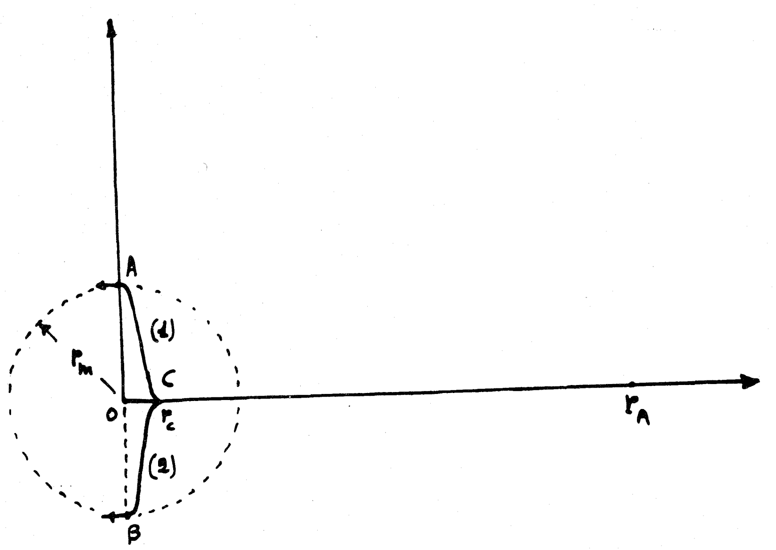

Fig. 4. – Qualitative sketch of the trajectories of two electrically-charged particles. Obviously, the paths between C and A, C and B, are not in scale.

$= r_c$ it is a substantially rectilinear path, (b) between $r = r_c$, $\theta = 0$ and $r = r_m$, $\theta = \pi/2$ (point A) we have a connecting path till point A (on the circumference of radius r_m). After this point the particle is inserted in circular motion according to Eq. (18). In the same way for a negative electrically-charged particle we will have the trajectory (2).

Let us go back to the particle electrization. If it is assumed that electric charge is uniformly distributed throughout its surface, we will have:

$$q = 4\pi r_0^2 \sigma_q \qquad m = \frac{4}{3} \pi r_0^3 \rho \tag{19}$$

here σ_q is the electric charge surface density. From Eqs. (19) we obtain

$$\frac{q}{m} = 3 \frac{\sigma_q}{\rho} \frac{1}{r_0}$$

Supposing $\sigma_q =$ const. and independent of r_0, the ratio q/m is proportional to r_0^{-1} as well as r_m proportional to $r_0^{-2/3}$. Table I collects the values r_m/r_{m_0} (being $r_{m_0} = 3.5 \times 10^8\ m$ as calculated before) versus the values of r_0.

TABLE I.

r_m/r_0	r_0 (μ m)
1.59	1
1	2
0.763	3
0.63	4
0.543	5

From this Table we can conclude that solid particles, having r_0 in the $1 - 5\ \mu$m range or smaller, should be collected in the neighbourhood of the first five Jovian satellites. Such a mechanism of accretion of " dusty plasma " seems to be reasonable.

Now we calculate the final velocity by means of formula (18). Its numerical value is

$$v_T = 2.7 \times 10^4 \text{ m } s^{-1} ,$$

while the circular keplerian velocity is

$$v_c = \sqrt{\frac{GM_p}{r_m}} = 1.89 \times 10^4 \text{ m } s^{-1} < v_T .$$

Let us consider two particles (with opposite electric charges and equal mass m) that fall towards Jupiter at two slightly different epochs, on the trajectories shown in fig. 4. As soon as they enter in a circular orbit, they will experience an encounter (or collision). At the instant of impact the opposite electric charges will be cancelled, allowing the recombination process. Furthermore, as a consequence of the impact, owing to the particular motion of two particles and to conservation of the momentum, in the instant after collision they will have equal but opposite velocity. Consequently, their orbits will be elliptic, with apocentre $r_1 > r_m$. If there were some dissipations of energy in the impact process, the apocentre should be $r_1' < r_1$.

The net result, however, will be the formation of a thin dusty disc, in dynamical equilibrium, around the planet Jupiter (on its equatorial plane).

5. CONCLUSIVE REMARKS

The only conjectural figure is the electric charge q of the particle. The criterion for the choice was the following one: the unit charge, one coulomb, is a large charge to be found in nature; let us take $1 \text{ m C} = 10^{-3}$ C and suppose it uniformly distributed on a spherical surface of radius $r_0 = 1$ m. If we wish to maintain the same σ_q on a spherical surface of radius $r_0 = 1$ μm, the net charge on these small particles will be of the order of 10^{-15} C.

Furthermore, an additional assumption was that the magnetic field emanating from the planet is a dipole field, with the dipole located at the centre of the planet and directed parallel to its spin axis.

REFERENCES

[1] D.A. MENDIS, H.L.F. HOUPIS and J.R. HILL (1982) – *The gravitoelectrodynamics of charged dust in planetary magnetospheres* « J. Geophys. Res. », *87*, 3449-3455.
[2] T.G. NORTHROP and J.R. HILL (1983) – *The adiabatic motion of charged dust grains in rotating magnetospheres*, « J. Geophys. Res. », *88*, 1-11.
[3] H. ALFVÉN and C. ARRHENIUS (1975) – *Structure and evolutionary history of the Solar System*, D. Reidel Publ. Co.
[4] J.D. KRAUS (1953) – *Electromagnetics*, McGraw-Hill Book Co.
[5] S.F. DERMOTT (1978) – *The origin of the Solar System*, J. Wiley and sons.
[6] B. LEHNERT (1964) – *Dynamics of charged particles*, North Holland Publ. Co.