ALDO BRESSAN

Some chain rules for certain derivatives of double tensors depending on other such tensors and some point variables. I. On the pseudo-total derivative


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Fisica matematica. — Some chain rules for certain derivatives of double tensors depending on other such tensors and some point variables. I. On the pseudo-total derivative. Nota (*) del Corrisp. ALDO BRESSAN.

RIASSUNTO. — Si considerano due spazi $S_\mu$ e $S_\sigma$, Riemanniani e a metrica eventualmente indefinita, riferiti a sistemi di co-ordinate $\varphi$ e $\varphi^*$; e inoltre un doppio tensore $T_{\mu\sigma}$ associato ai punti $\varphi^{-1}(x) \in S_\mu$ e $\varphi^*-1(y) \in S_\sigma$. Si pensa $T_{\mu\sigma}$ dato da una funzione di $m$ altri tali doppi tensori e di variabili puntuali $x(\in \mathbb{R}^\mu), t \in \mathbb{R}$ e $y(\in \mathbb{R}^\sigma)$; poi si considera la funzione composta

$$T_{\mu\sigma}(x, t, y) = T_{\mu\sigma}[H_{\nu\rho}(x, t, y), \ldots, H_m(x, t, y), x, t, y].$$

Nella Parte I si scrivono due regole per eseguire la derivazione totale di questa, connessa con una mappa $\varphi = \varphi(t)$ fra $S_\mu$ e $S_\sigma$; una è a termini generalmente non covarianti e l'altra a termini (sempre) covarianti. Si applicano queste regole per esprimere il risultante $P$ degli sforzi in un corpo (iper-)elastico classico.

Nella Parte II si scrivono due regole analoghe per la derivata assoluta di $T_{\mu\sigma}$, e altre due per la derivata Lagrangiana spaziale (o trasversa) $T_{\mu\sigma}$ di $T_{\mu\sigma}$. La $T_{\mu\sigma}$ è utile in Relatività generale o ristretta; e si applicano le due regole riferentesi ad essa per scrivere due espressioni di $P$ appunto nel caso di un corpo (iper-)elastico relativistico.

§ 1. INTRODUCTION (***)

The total derivative $T_{\mu\sigma}$ of a double tensor field $T_{\mu\sigma}(x, t, y)$ where $x, t, y$ are point variables—see (2.8) or [3]—is used also in classical physics, for instance, to treat continuous media in general co-ordinates—see e.g. [4]—. In general relativity, where everywhere (pseudo-) Euclidean co-ordinates are lacking, algorithms enabling us to use general co-ordinates are more important than in classical physics. However, in this theory a natural representation of the motion $\mathcal{M}$ of a continuous body $\mathcal{E}$ depends on an arbitrary function $\tilde{t}$ (time parameter)—see N 6, or better § 52 in [2]. Therefore in [1]

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(***) The present work, performed in the sphere of activity of research group n. 3 of the Consiglio Nazionale delle Ricerche, in 1984 and 1985, is an improved and enriched version of some lessons given by the author in his course of Continuum Mechanics (Padua 1984-85) and in his CIME course of non-stationary relativistic thermodynamics (Ravello Sept. 1985).
Some chain rules for certain derivatives,

For instance, let $\mathcal{C}'$ be a possibly non-homogeneous hyperelastic body within classical physics [special or general relativity]. Then the 1st Piola Kirchhoff stress tensor $K^a_b$ is expressed by a constitutive function $\tilde{K}^{ab}$ whose arguments are other double tensors, say $H^{1a}$ to $H^{mb}$ and some point variables, such as the set $y$ of the reference co-ordinates $y^1, y^2, y^3$ of the typical matter point $P_0$ of $\mathcal{C}'$—see (5.1), (10.1). Along the motion $\mathcal{M}$, represented by the equation $x = x(t,y)$, we have $H^{ia} = H(t,y)$. Then, in order to calculate the spatial stress divergence, we have to calculate the pseudo-total Lagrangian spatial derivative of a compound function, whose form is included in the form

\begin{equation}
T^{ia}(x,t,y) = T^{ia}(H^{1a}, \ldots, H^{mb}, x, t, y, x, t, y),
\end{equation}

where $x$ is the set of co-ordinates for the actual position of $P_0$ in the kinematic space being considered (in space time) (2).

In order to calculate the derivatives $T^{ia}(x,t,y);_R$, $T^{ia}(x,t,y),_R$, and the absolute (relativistic) derivative $DT^{ia}(x,t,y)/D_s$, chain rules are not strictly necessary; however, they are useful. Therefore, in this work two chain rules are stated for each of the three derivatives above, one with generally non-convariant terms and the other with only covariant terms—see (3.5), (4.8), (9.1), (9.3), (9.5), and (9.7). In the relativistic case the terms of the latter rule are also independent of the choice of $\hat{t}$. Furthermore a certain equality which in my opinion has some chances of being taken as a natural chain rule—see e.g. (3.6)—is shown to be generally false, unless both co-ordinate systems being used are locally geodesic.

As examples, the rules for $T^{ia};_R [T^{ia};]_R$ are used to calculate the density $F$ of the local internal forces for $\mathcal{C}'$ in classical physics [N 5] and relativity theory [N 10].

This work consists of two notes: Part 1 and Part 2. The former is devoted to $T^{ia};_R$ and classical physics whereas the latter is mainly concerned with $DT^{ia}/D_s, T^{ia};_R$, and relativity theory.

In the typical case the derivative $T^{ia}(H^{1a}, \ldots, H^{mb}, x, t, y);_R$ involves partial derivatives of $T^{ia}$ with respect to only a part of $T^{ia}$'s arguments. Therefore it is called pseudo-total derivative. Also the pseudo-absolute derivative $DT^{ia}/D_p$ of $T^{ia}$ is considered, i.e. the absolute derivative of $(x, t, y)$

(1) In [1] and [2] I called $\tilde{T}^{ia};_R Lagrangian transverse derivative of $T^{ia}$. However the qualification spatial seems to me now more appropriate than transverse.

(2) The constitutive function $\tilde{K}^{ab}$ for $K^{ab}$ must also have a time parameter $t$ as an argument, in case $C'$ is undergoing some chemical reactions independent of $\mathcal{M}$.
For $m > 0$ it generally fails to be covariant as well as $\tilde{T}^{\alpha\beta\cdots}_{\cdots\alpha\beta}$ and $\tilde{T}^{\alpha\beta\cdots}_{\cdots\alpha\beta}_{\check{R}}$. Therefore the stationary (or covariant partial) pseudo-total derivative $\tilde{T}^{\alpha\beta\cdots}_{\cdots\alpha\beta}_{\check{S}_{\check{R}}}$ of $\tilde{T}^{\alpha\beta\cdots}_{\cdots\alpha\beta}$ is introduced [§4] and the analogue is done with $D \tilde{T}^{\alpha\beta\cdots}_{\cdots\alpha\beta}$ and $\tilde{T}^{\alpha\beta\cdots}_{\cdots\alpha\beta}_{\check{R}}$ [§9]. These stationary derivatives enter the chain rules, all of whose terms are covariant. The remaining chain rules involve connectionless derivatives—see (3.4), (48.), and (8, 1-2).

\section{Double Tensors and Total Derivatives}

Let $S^\mu$ and $S^*_\nu$ be Riemannian spaces of respective dimensions $\mu$ and $\nu$. Their metric tensors $g_{\alpha\beta}$ and $a^*_{LM}$ may fail to be defined $> 0$ (strictly positive) or $< 0$ (strictly negative), and may also be everywhere Euclidean or everywhere pseudo-Euclidean.

Let $\{x^a, \gamma\}$ and $\{x^a, \gamma\} = \{x^a, \gamma\} g^{\alpha\beta}$, where $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, be the Christoffel symbols for $S^\mu$ and let $\{A, B, C\}$ and $\{^{R}_{AB}\}$ be their analogues for $S^*_\nu$.

Consider the points $\mathcal{S} \in S^\mu$ and $\mathcal{P} = S^*_\nu$; and let $\phi\{a^*\}$ be a (regular) frame, or coordinate system, for $S^\mu$ [$S^*_\nu$], i.e. a bijection of $S^\mu$ [$S^*_\nu$] onto an open subset of $R^\mu$ [$R^\nu$], e.g.

\begin{equation}
(x^1, \ldots, x^\mu) = \phi(\mathcal{S}) = (\phi^1(\mathcal{S}), \ldots, \phi^\mu(\mathcal{S}))(\forall \mathcal{S} \in S^\mu);
\end{equation}

(2.1)

\begin{equation}
(y^1, \ldots, y^\nu) = \phi^*(\mathcal{P}^*)(\forall \mathcal{P}^* \in S^*_\nu).
\end{equation}

Frame $\phi\{a^*\}$ can also be denoted by $(x)\{[y]\}$. Now consider the set of $\mu^{a+c} \cdot \nu^{b+d}$ scalars

\begin{equation}
\{T_{a_1 \cdots a_n B_1 \cdots B_d S_1 \cdots S_d}\} = T(\phi, \phi^*),
\end{equation}

where Greek [Latin] indices run over a set of $\mu$ [$\nu$] elements—e.g. from 1 to $\mu$ [$\nu$]. Let it depend on $\phi$ and $\phi^*$ in such a way that, whenever also $\hat{\phi}\{a^*\}$ is a frame for $S^\mu$ [$S^*_\nu$], we have

\begin{equation}
\tilde{T}_{a_1 \cdots a_n B_1 \cdots B_d} = T_{a_1 \cdots a_n B_1 \cdots B_d} \frac{\partial x^1}{\partial x^{a_1}} \cdots \frac{\partial x^a}{\partial x^{a_n}} \cdots \frac{\partial y^1}{\partial y^{B_1}} \cdots \frac{\partial y^B}{\partial y^{B_d}} \cdots,
\end{equation}

where (i) $T_{a_1 \cdots a_n B_1 \cdots B_d} = T(\hat{\phi}, \hat{\phi}^*)$, (ii) $\partial x^a / \partial x^b$ are the partial derivatives of the function $\frac{\partial x^a}{\partial x^b} = \frac{\partial x^a}{\partial x^b}(x^1, \ldots, x^\mu)$ evaluated at the point $\phi(\mathcal{S})$ of $R^\mu$, and (iii) the analogues hold for $\partial y^b / \partial y^c$, $\partial y^R / \partial y^A$, and $\partial y^B / \partial y^S$. Then $T$ is said to be a (double) tensor of covariant order $(a, c)$ and contravariant order $(b, d)$.

(3) If one likes to consider an atlas for e.g. $S^\mu$, let $\phi$ be a bijection of an open subset of $S^\mu$ that includes $\mathcal{S}'$, into an open subset of $R^\mu$. 

\textbf{Abbreviations:}

- $\mathcal{F}$: Function
- $\mathcal{S}$: Space
- $\mathcal{P}$: Point
- $\mathcal{S}'$: Subset
- $\mathcal{S}^*$: Star

\textbf{References:}

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(b, d), attached to the point $\mathcal{E}$ of $S_\mu$ through its first $a + b$ indices, and to the point $P^*$ of $S^*$ through its last $c + d$ indices. The scalars $T^{a_1...A_1...} = T^{a_1...S_1...}$ — see (2.2) — are called the components of $T$ in frames $\phi$ and $\phi^*$. Now regard the above double tensor $T$ as a function $\bar{T}$ whose arguments are $m \geq 0$ other double tensors $H$ to $H$ also attached to $\mathcal{E}$ and $P^*$, the point variables $\mathcal{E}$ and $P^*$, and (possibly) a real parameter $t$. Let all arguments of $\bar{T}$ range over some open subsets of some suitable spaces.

Let us remark that the above parameter $t$ is used throughout Part 1 mainly for purposes reached in Part 2. Readers interested in (pseudo-) total derivatives but not in (pseudo-) absolute or Lagrangian spatial derivatives, can cross out $t$ everywhere in §§ 2-4.

Field $\bar{T}^{a_1...A_1...}$ is represented in the above frames $\phi$ and $\phi^*$ by the component functions

\[(2.4) \quad T^{a_1...A_1...} = \bar{T}^{a_1...A_1...} (H^{a_1...A_1...}, \ldots, H^{a_1...A_1...}, x, t, y).\]

Let us now define the pseudo-covariant partial derivatives (of $\bar{T}$) in $S_\mu$ and $S^*$, by

\[(2.5) \quad \frac{\partial \bar{T}^{a_1...A_1...}}{\partial x^\rho} = St_{\rho} \bar{T}^{a_1...A_1...}, \quad \frac{\partial \bar{T}^{a_1...A_1...}}{\partial y^R} = St_{R} \bar{T}^{a_1...A_1...},\]

where (the linear operators) $St_{\rho}$ and $St_{R}$ are given by

\[(2.6) \quad \begin{cases} 
St_{\rho} T^{a_1...A_1...} = \{ S_{a_1} \} T^{a_1...A_1...} + \ldots + \{ S_{A_1} \} T^{a_1...A_1...} - \ldots, \\
St_{R} T^{a_1...A_1...} = \{ R_{A_1} \} T^{a_1...A_1...} + \ldots + \{ R_{A_1} \} T^{a_1...A_1...} + \ldots.
\end{cases}\]

The symbols thus introduced are justified by simple stationarity properties — see below (4.5).

For $m = 0$ each of the scalar systems (2.5) (which depend on $\phi$ and $\phi^*$) turns out to be a double tensor attached to $\mathcal{E} \in S_\mu$ and $P^* \in S^*$. Hence one speaks of covariant partial derivatives of the double tensor field $\bar{T}^{a_1...A_1...}$ (regarded, if preferred, as a function of $x$ and $y$).

Consider a $C^{(1)}$-homeomorphism $\mathcal{E} = \hat{\mathcal{E}} (P^*)$ of $S_\mu$ into $S^*$, possibly depending on $t (\mathcal{E} = \hat{\mathcal{E}}_t)$ and represented, in frames $\phi$ and $\phi^*$, by

\[(2.7) \quad x^\rho = \hat{x}^\rho (t, y) — or precisely \quad x^\rho = \hat{x}^\rho (y) = \hat{x}^\rho (t, y) (\mathcal{E} = \hat{\mathcal{E}}_t, \hat{x}^\rho \in C^{(1)}).\]

For any $m \geq 0$, the pseudo-total derivative of the field $\bar{T}^{a_1...A_1...}$ — see (2.4) — connected with this map, is defined by

\[(2.8) \quad \bar{T}^{a_1...A_1...} = \bar{T}^{a_1...A_1...} / x^\rho + \bar{T}^{a_1...A_1...} / y^R \quad (x^\rho = \partial \hat{x}^\rho / \partial y^R).\]
For $m = 0$, it is also a double tensor attached to $\mathcal{E}$ and $P^*$, but dependent on only $P^*$ (and $t$). Some of its properties can be found in [3]—see also [2], p. 234.

§ 3. Compound functions whose arguments involve double tensors and point variables. A chain rule for the total derivative of them, with generally non covariant terms

Besides field (2.4), consider the $m$ double tensor fields

$$ (3.1) \quad H^{i_1 \ldots i_m}_{\ldots L \ldots} = \bar{H}^{i_1 \ldots i_m}_{\ldots i} (x, t, y) \quad (i = 1, \ldots, m); $$

and remembering (2.7), let us set

$$ (3.2) \quad T^{a_1 \ldots a_m}_{s_1 \ldots s_m} = \bar{T}^{a_1 \ldots a_m}_{s_1 \ldots s_m} (x, t, y) = \bar{T}^{\ldots}_{a_1 \ldots a_m}(H^{i_1 \ldots i_m}_{\ldots i}(x, t, y), \ldots, m). $$

Then

$$ (3.3) \quad \bar{T}^{\ldots}_{a_1 \ldots a_m} \sim R = \sum_{i=1}^{m} \frac{\partial \bar{T}^{\ldots}_{a_1 \ldots a_m}}{\partial H^{i_1 \ldots i_m}_{\ldots i}} \bar{H}^{i_1 \ldots i_m}_{\ldots i} \sim R \sim R + \bar{T}^{\ldots}_{a_1 \ldots a_m} \sim R $$

where, for an arbitrary choice of the field $\bar{T}^{\ldots}_{a_1 \ldots a_m}(m \geq 0)$, its connectionless pseudo-total derivative $\bar{\bar{T}}^{\ldots}_{a_1 \ldots a_m} \sim R$ is defined by—see (2.8)—

$$ (3.4) \quad \bar{T}^{\ldots}_{a_1 \ldots a_m} \sim R = \frac{\partial T^{\ldots}_{a_1 \ldots a_m}}{\partial x^p} x_p^R + \frac{\partial T^{\ldots}_{a_1 \ldots a_m}}{\partial y^R} \frac{\partial y^R}{\partial y^R} , $$

hence

$$ (3.4') \quad \bar{\bar{T}}^{\ldots}_{a_1 \ldots a_m} \sim R \sim R \text{ for } g_{a_b, c} = 0 = a_{AB}^* \left( f_p = \frac{\partial f}{\partial x^p}, f_R = \frac{\partial f}{\partial y^R} \right). $$

Expression (3.4) of $\bar{\bar{T}}^{\ldots}_{a_1 \ldots a_m} \sim R$ can be obtained from the one of $T^{\ldots}_{a_1 \ldots a_m} \sim R$—see (2.8) and (2.5-6)—by crossing out the terms in the connections $\{x_p^R\}$ and $\{C_{AB}\}^*$, which justifies the name for $\bar{\bar{T}}^{\ldots}_{a_1 \ldots a_m} \sim R$. An analogous use of a redoubled derivation sign will be made in Part 2—see (8.2).

By (2.8), (2.5-6), and (3.3) we easily deduce the chain rule

$$ (3.5) \quad \bar{T}^{\ldots}_{a_1 \ldots a_m} (x, t, y) \sim R = \sum_{i=1}^{m} \frac{\partial \bar{T}^{\ldots}_{a_1 \ldots a_m}}{\partial H^{i_1 \ldots i_m}_{\ldots i}} \bar{H}^{i_1 \ldots i_m}_{\ldots i} \sim R \sim R + \bar{T}^{\ldots}_{a_1 \ldots a_m} \sim R $$

for compound functions such as (3.2). For $m > 0$ its last two terms are generally
non-covariant. In fact $\tilde{T}^{\iota,\rho}_{\iota,\rho}(x,t,y)\cdot R$ and $\partial\tilde{T}^{\iota,\rho}_{\iota,\rho}/\partial H^{\iota,\rho}_{\iota,\rho}$ are covariant ($i=1,\ldots,m$), while $H^{\iota,\rho}_{\iota,\rho}$; $R$ generally fails to be so. Let us remark explicitly that therefore $\tilde{T}^{\iota,\rho}_{\iota,\rho}$ generally fails to be covariant when $m > 0$.

By (3.5) the inequality

$$\tilde{T}^{\iota,\rho}_{\iota,\rho}(x,t,y)\cdot R \neq \sum_{i=1}^{m} \frac{\partial\tilde{T}^{\iota,\rho}_{\iota,\rho}}{\partial H^{\iota,\rho}_{\iota,\rho}} \cdot R + \tilde{T}^{\iota,\rho}_{\iota,\rho} \cdot R$$

—see (3.2)—, obviously holds in the typical case. I note this because it seems to me relatively natural to assert the equality of the two sides of (3.6), in that the noncovariant character of $\tilde{T}^{\iota,\rho}_{\iota,\rho}$; $R$ may be overlooked. This equality is acceptable after the replacement of its last term by a suitable co-variant one—see (4.8).

§ 4. STATIONARY PSEUDO-TOTAL DERIVATIVES FOR FUNCTIONS SUCH AS (2.4). A CHAIN RULE FOR THE PRECEDING COMPOUNDS FUNCTIONS, ALL OF WHOSE TERMS ARE COVARIANT

Let us first assume that spaces $S^\mu$ and $S^\ast_\nu$ are (pseudo-) Euclidean, so that some choices of $\phi$ and $\phi^\ast$ render (3.4')_\varphi a, true everywhere. Then, if the component functions $H^{\iota,\rho}_{\iota,\rho}$ to $H^{\iota,\rho}_{\iota,\rho}$ are constant, they represent constant double tensor fields. Thus the tensors $T^{\iota,\rho}_{\iota,\rho}$ and $H^{\iota,\rho}_{\iota,\rho}$ to $H^{\iota,\rho}_{\iota,\rho}$ in (2.4) can be regarded as attached simply to $S^\mu$ and $S^\ast_\nu$. Hence for $H$ to $H$ fixed, $T^{\iota,\rho}_{\iota,\rho}$ can be regarded as a double tensor of $S^\mu$ and $S^\ast_\nu$ depending on $x$ and $y$ (and $t$). For the resulting field $\tilde{T}^{\iota,\rho}_{\iota,\rho}(x,y,t)$ we have

$$(4.1) \quad \tilde{T}^{\iota,\rho}_{\iota,\rho} \cdot R = \tilde{T}^{\iota,\rho}_{\iota,\rho}(x,t,y)\cdot R \quad \text{for} \quad g_{\alpha\beta,\gamma} = 0 = a^\ast_\alpha_{AB,C}.$$

Now let $S^\mu$ and $S^\ast_\nu$ be arbitrary Riemannian spaces, so that constant double tensor fields of many orders fail to exist in them. Therefore (4.1) can be considered only locally, by choosing $H^{\iota,\rho}_{\iota,\rho}$ locally stationary:

$$\tilde{H}^{\iota,\rho}_{\iota,\rho} \cdot R = 0 = \tilde{H}^{\iota,\rho}_{\iota,\rho} \cdot R, \quad \text{or at least} \quad \tilde{H}^{\iota,\rho}_{\iota,\rho} \cdot R = 0 \quad \text{for} \quad i = 1, \ldots, m.$$

With a view to writing the chain rule hinted at in the title, for arbitrary choices of $\phi$ and $\phi^\ast$, let us continue the considerations about (4.1) as follows. Fix arbitrary local values for the arguments $H^{\iota,\rho}_{\iota,\rho}$ to $H^{\iota,\rho}_{\iota,\rho}$ of function (2.4), attached to $P^\ast = \phi^{-1}(y) \in S^\ast_\nu$ and $\delta = \delta^\ast(P^\ast) = \phi^{-1}[\tilde{x}(t,y)] \in S^\mu$—see
Furthermore consider arbitrary tensor fields $\tilde{H}_i^{:::}(x', t, y')$ attached to $y'$ and $x(t, y')$ ($i = 1, \ldots, m$), that at $y$ and $\tilde{x}(t, y)$ (i) assume the locally fixed values and (ii) are pseudo-totally stationary:

$$\tilde{H}_i^{:::}(x, t, y) = H_i^{:::}, \quad \tilde{H}_i^{:::}(x, t, y)_R = 0 \quad (i = 1, \ldots, m).$$

Such tensor fields certainly exist, even with $\partial \tilde{H}_i^{:::}/\partial x^p \equiv 0 \equiv \partial H_i^{:::}/\partial t$.

In fact

$$\tilde{H}_i^{:::;f}_R = \tilde{H}_i^{:::;f}_R + St_{tR} H_i^{:::} \quad \text{for} \quad H_i^{:::} = \tilde{H}_i^{:::}, [\tilde{x}(t, y), t, y]$$

where

$$St_{tR} H_i^{:::} = (St_v, H_i^{:::}) x_R^f + St_{tR} H_i^{:::}$$

—see (2.6).

Therefore (4.2)$_a$ is equivalent to

$$\tilde{H}_i^{:::;f}_R = St_{tR} H_i^{:::} \quad (\tilde{H}_i^{:::;f}_R = \tilde{H}_i^{:::;f}_R \text{ for } \tilde{H}_i^{:::;f}_R \text{ if } v = 0).$$

Thus $St_{tR} H_i^{:::}$ is the connectionless pseudo-total derivative of a stationary field of local value $H_i^{:::}$. Incidentally $St_v H_i^{:::} [St_{tR} H_i^{:::}]$ is its analogue for the partial pseudo-covariant derivative in $S_a [S_v^a]$.

Remembering (3.2) and (4.2) one can now define the stationary, or covariant partial, pseudo-total derivative $\tilde{T}_i^{:::} = St_{tR} T_i^{:::}$ (connected with the map $\tilde{\sigma}$):

$$\tilde{T}_i^{:::;f}_R = D \tilde{T}_i^{:::;f}_R (x, t, y)_R \quad \text{for} \quad \tilde{H}_i^{:::;f}_R = \tilde{H}_i^{:::;f}_R \quad (i = 1, \ldots, m).$$

Then (4.5) and (3.5) yield the explicit expression

$$\tilde{T}_i^{:::;f}_R (H_i^{:::}, \ldots, H_m^{:::}, x, t, y) = \sum_{i=1}^m \frac{\partial \tilde{T}_i^{:::;f}_R}{\partial H_i^{:::}} St_{tR} H_i^{:::} + \tilde{T}_i^{:::;f}_R.$$

By (4.7) and (4.3), (3.5) yields the chain rule

$$\tilde{T}_i^{:::;f}_R (x, t, y)_R = \sum_{i=1}^m \frac{\partial \tilde{T}_i^{:::;f}_R}{\partial H_i^{:::;f}} \tilde{H}_i^{:::;f} + \tilde{T}_i^{:::;f}_R \text{ for compound functions such as (3.2), all of whose terms are covariant.}$$
§ 5. THE STRESS DIVERGENCE FOR A HYPERELASTIC BODY AND THE ABOVE CHAIN RULES

Identify $S_{3}$ and $S_{3}^{*}$ with a same inertial space. Furthermore assume that (i) $\mathcal{C}$ is a hyperelastic—i.e. a purely mechanical elastic-body, (ii) $C^{*}$ is a reference configuration for it, regarded as belonging to $S_{3}^{*}$, (iii) any motion $\mathcal{M}$, possible for $\mathcal{C}$, is represented by an equation such as (2.7)1 with $\tilde{x} \in C^{(2)}$, so that $\mathcal{C}$ can be thought of as a set of material points, and (iv) $P^{*}$ is a typical one among these points.

Then (using the above frames $\phi$ and $\phi^{*}$), at any instant $t$, the first Piola-Kirchhoff stress tensor $K^{aB}$ at $P^{*}$ in $\mathcal{M}$, is a double tensor attached to $P^{*}$ and $\tilde{C}^{*}(P^{*})$ through the indices $a$ and $B$ respectively; furthermore it is given by a constitutive equation of the form (4)

\begin{equation}
K^{aB} = \tilde{K}^{aB}(y, x^{\mu}_{L}, g_{\lambda \mu}, \phi^{*}) \quad \text{where} \quad y = (y^{1}, y^{2}, y^{3}) = \phi^{*}(P^{*}).
\end{equation}

Along $\mathcal{M}$ (5.1) induces a function $K^{aB} = \tilde{K}^{aB}(t, y)$ in the well known way.

The dynamic equations for $\mathcal{C}$ involve the resultant $I^{a} = -\tilde{K}^{aB}_{;B}$ of the internal forces acting on $\mathcal{C}$ at $P^{*}$, per unit reference volume. By rules (3.5) and (4.8) with $H^{L}_{\lambda} = x^{\lambda}_{L}(t, y)$ and $H_{\lambda \mu} = g_{\lambda \mu}(x)$ $I^{a}$ has the expressions

\begin{equation}
I^{a} = -\tilde{K}^{aB}_{;B} x^{\mu}_{L} = \frac{\partial \tilde{K}^{aB}}{\partial x^{\mu}_{L}} x^{\mu}_{L} - \frac{\partial K^{B}}{\partial g_{\lambda \mu}} g_{\lambda \rho} \frac{\partial x^{\rho}_{B}}{\partial x^{L}_{B}} - \tilde{K}^{aB}_{,B}(x^{\mu}_{L} = x^{\mu}_{L})
\end{equation}

and

\begin{equation}
I^{*} = -\frac{\partial \tilde{K}^{aB}}{\partial x^{L}_{L}} x^{\mu}_{L} = \tilde{K}^{aB}_{,B} x^{\mu}_{L} - \tilde{K}^{aB}_{,B}(x^{\mu}_{L} = x^{\mu}_{L})
\end{equation}

respectively, where by (4.7) and (4.4)

\begin{equation}
\tilde{K}^{aB}_{,B} = \tilde{K}^{aB}_{;B} - \frac{\partial K^{aB}}{\partial x^{\mu}_{L}} (\delta_{\sigma}^{a}) x^{\sigma}_{L} x^{B}_{B} - (\delta_{B}^{a}) x^{\mu}_{L} x^{3}_{3}.
\end{equation}

Incidentally, since $S_{3} = S_{3}^{*}$, one can choose $\phi = \phi^{*}$. However also in this case $\{ \}$ and $\{ \}^{*}$ are generally unrelated, because they are calculated at different points.

(4) $\tilde{K}^{aB}$ behaves in the obvious way under changes of $\mathcal{C}$, and it is determined by the function induced by it for any particular choice $\mathcal{Z}^{*}$ of $\mathcal{C}$ and for $g_{\mu \nu} = \delta_{\mu \nu}$. In more detail, let $\tilde{y} = \tilde{y}(y)$ be $\mathcal{Z}^{*} \circ \mathcal{C}^{*} - 1$, let $(x_{3}^{\sigma})$ be any matrix for which $g_{\lambda \mu} = \delta_{\gamma \delta} x_{3}^{\gamma} x_{3}^{\delta}$, and set $(x_{3}^{\sigma}) = (x_{3}^{\sigma})^{-1}$. Then (10.1) holds if and only if

\begin{equation}
K^{aB} = x_{3}^{\sigma} \left( \partial_{y^{B}} / \partial y^{S} \right) \mathcal{Z}^{*} \left( \tilde{y}, \tilde{x}_{3}^{\sigma}, g_{\lambda \mu}, \mathcal{Z}^{*} \right) \quad \text{with} \quad \tilde{a}^{3}_{R} = a_{L}^{3} x_{3}^{\gamma} \partial y^{A} / \partial y^{R}.
\end{equation}
Let us add that in accordance with inequality (3.6), by (5.3-4) the equality

\[
I^a = - \frac{\partial K^{aB}}{\partial x^L} x^a_{;LB} - \bar{K}^{aB}_{;B}
\]

is generally false. It is true and coincides with both (5.2) and (5.3) in locally geodesic co-ordinates \((g_{ab}, \lambda = 0 = a^*_{AB,C})\).

REFERENCES