Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Gianfranco Cimmino

An unusual way of solving linear systems

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **80** (1986), n.1-2, p. 6–7.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1986_8_80_1-2_6_0>

Analisi matematica. — *An unusual way of solving linear systems.* Nota (*) del Socio GIANFRANCO CIMMINO.

RIASSUNTO. — Mediante integrali multipli agevoli per il calcolo numerico vengono espressi il valore assoluto di un determinante qualsiasi e le formule di Cramer.

Let $x = (x_1, \dots, x_n)$ be the solution of the linear system

$$\sum_1^n {}_k\, a_{hk}\, x_k = b_h\,, \quad h = 1, \dots, n\,, \qquad \det(a_{hk}) \neq 0\,. \tag{1}$$

The $\sum_1^n {}_k$ at the left-hand side will be considered as the scalar product of the vectors $a_h = (a_{h1}, \dots, a_{hn})$, x in $\mathbf{R}^n$.

One finds the following identities to hold

$$|\det(a_{hk})| = |\omega| \left[\int_\omega \left\| \sum_1^n {}_h\, \xi_h\, a_h \right\|^{-n} d\omega \right]^{-1}, \tag{2}$$

$$x = \frac{n}{|\omega|} \left|\det(a_{hk})\right| \int_\omega \left\| \sum_1^n {}_h\, \xi_h\, a_h \right\|^{-n-2} \sum_1^n {}_h\, \xi_h\, a_h \sum_1^n {}_k\, \xi_k\, b_k\, d\omega\,, \tag{3}$$

where $|\omega|$ indicates the $(n-1)$-dimensional measure of the spherical variety

$$\omega = \left\{ \xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n \text{ s. t. } \|\xi\|^2 = \sum_1^n {}_h\, \xi_h^2 = 1 \right\}, \tag{4}$$

i.e.

$$\omega = \frac{(\sqrt{2\pi})^{n-1}}{(n-2)!!} \begin{cases} \sqrt{2\pi}\,, & n \text{ even,} \\ 2\,, & n \text{ odd,} \end{cases} \tag{5}$$

and $d\omega$ is the corresponding $(n-1)$-dimensional measure element.

(*) Presentata nella seduta dell'11 gennaio 1986.

I didn't try to check whether formulas (2),(3) are already known, as it seems rather likely. Anyhow it may perhaps be of some interest to see how I got them.

In $\mathbf{R}^n$ the points y symmetric of the origin 0 respect to hyperplanes containing the unknown x of (1) are all at the same distance $\| y - x \| = \| x \|$ from x itself. And as many of such points y as one likes are clearly available by taking arbitrary linear combinations of the equations (1).

The centre of gravity $\tilde{y}$ of a system of masses concentrated at these points y will have a distance $\| \tilde{y} - x \|$ from x in any case less than $\| x \|$.

Furthermore, basing on a probabilistic argument, one can expect the probability of getting a distance $\| \tilde{y} - x \|$ smaller than a given $\theta \| x \|$, $0 < \theta < 1$ to grow with the number of the considered hyperplanes, as I remarked in a preceding paper of mine (1) in which I also suggested considering some continuous families of these hyperplanes, and consequently continuous distributions of masses on the sphere $\| y - x \| = \| x \|$.

Formulas (2), (3) arise now from the attempt to find a continuous distribution of suitable masses, in correspondence to which the distance $\| \tilde{y} - x \|$ reduces to zero.

They can then of course be also directly proved.

(1) « Su uno speciale tipo di metodi probabilistici in analisi numerica », Symposia Mathematica, X (1972), 247-254.