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About some parameters of normed linear spaces

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Analisi funzionale. — *About some parameters of normed linear spaces.* Nota di EMANUELE CASINI, presentata (*) dal Corrisp. R. CONTI.

RIASSUNTO. — Si prendono in considerazione particolari costanti relative alla struttura della sfera unitaria di uno spazio di Banach. Se ne studiano alcune generali proprietà, con particolare riferimento alle relazioni con il modulo di convessità dello spazio. Se ne fornisce inoltre una esatta valutazione negli spazi l_p.

1. INTRODUCTION AND NOTATIONS

Let X be a Banach space over the real field.

We denote by X′ the dual of X and by S (X) its unit sphere, i.e. $S(X) = \{x \in X : \|x\| = 1\}$.

In [4], Gao defined the following constants for X:

$$g(X) = \inf_{x \in S(X)} \{ \inf_{y \in S(X)} [\max(\|x-y\|, \|x+y\|)] \}$$

$$G(X) = \sup_{x \in S(X)} \{ \inf_{y \in S(X)} [\max(\|x-y\|, \|x+y\|)] \}$$

$$g'(X) = \inf_{x \in S(X)} \{ \sup_{y \in S(X)} [\min(\|x-y\|, \|x+y\|)] \}$$

$$G'(X) = \sup_{x \in S(X)} \{ \sup_{y \in S(X)} [\min(\|x-y\|, \|x+y\|)] \}$$

The value of these constants is known in some special cases.

For example, $g(L^1) = G(L^1) = 1$, $g'(L^1) = G'(L^1) = 2$; $g(l_1) = 1$, $G(l_1) = g'(l_1) = G'(l_1) = 2$; $g(l_\infty) = g'(l_\infty) = 1$, $G(l_\infty) = G'(l_\infty) = 2$. The four parameters are continuous with respect to the Banach-Mazur distance, in any class of isomorphic spaces.

For any space X we have

$$\begin{array}{ccccc} 1 \leq & g(X) & \leq & G(X) & \\ & \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & \\ & g'(X) & \leq & G'(X) & \leq 2 . \end{array}$$

For these and some related results we refer to [4] and [5] (see also [9]).

(*) Nella seduta dell'8 febbraio 1986.

Still in [9], it was noted that some relations exist between $g(X)$ and $G(X)$, and other constants related to the notion of girth (see [11]).

In Section 2 of the present note we want to relate some of Gao's constants to the moduli of convexity and smoothness of X, defined by

(1) $\delta_X(\varepsilon) = \frac{1}{2} \inf \{ 2 - \| x + y \| : x, y \in S(X) \text{ and } \| x - y \| = \varepsilon \} \quad 0 \le \varepsilon \le 2$

(2) $\rho_X(\tau) = \frac{1}{2} \sup \{ \| x + \tau y \| + \| x - \tau y \| - 2 : x \in S(X) ; \| y \| = \tau \} \quad 0 \le \tau$

Also, we give in Section 3 the exact value of the above constants when $X = l_p$.

2. General results

We say that X is uniformly non-square (UNS) if $\delta_X(\varepsilon) > 0$ for some $\varepsilon < 2$. Concerning the moduli of convexity and smoothness, we recall the following estimates; for any X we have:

$$\delta_X(\varepsilon) \le \delta_{l^2}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \tag{3}$$

$$2\rho_X'(\tau) = \sup_{\varepsilon > 0} (\tau\varepsilon - 2\,\delta_X(\varepsilon)) \tag{4}$$

The following modification of the modulus of convexity was defined by Milman (see [7, p. 84]):

$$\delta_1(\varepsilon, X) = \inf_{x, y \in S(X)} \{ \max(\| x - y \| \ , \ \| x + y \|) - 1 \} . \tag{5}$$

Clearly, we have

$$\delta_1(1, X) = g(X) - 1 . \tag{6}$$

As proved in [2], Lemma 6, the following relation holds:

$$(1 - \delta_X(\varepsilon))\, \delta_1(\varepsilon/2\,(1 - \delta_X(\varepsilon)), X) = \delta_X(\varepsilon) . \tag{7}$$

Note that still in [7] another constant, a modification of $\rho_X(\tau)$, was considered; unfortunately, the connection with $\rho_X(\varepsilon)$ indicated there is not correct (as pointed out in [2, p. 125]).

Inequality (7) implies the following

PROPOSITION 2.1. *If* $g(X) > 1$, *then* X *is* (UNS).

Conversely, let X *be* (UNS). *Then, if* $\bar{\varepsilon} \in (0, 2)$ *is the unique solution of*

$$\bar{\varepsilon} = 2(1 - \delta_X(\bar{\varepsilon})) \tag{8}$$

we have $g(X) = \dfrac{2}{\bar{\varepsilon}} \in (1, \sqrt{2}]$.

Proof. Let X be not (UNS). Then, given $\varepsilon > 0$, there exist u, v in $S(X)$ such that $\| u + v \| \geq 2 - \varepsilon$, $\| u - v \| \geq 2 - \varepsilon$. Now, if we set

$$x = \frac{u + v}{\| u + v \|} \quad \text{and} \quad y = \frac{u - v}{\| u - v \|},$$

we obtain

$$\| x \pm y \| \leq \frac{| \| u + v \| \pm \| u - v \| | + | \| u + v \| \mp \| u - v \| |}{\| u + v \| \quad \| u - v \|} \leq$$

$$\leq \frac{2(2 - \varepsilon) + \varepsilon}{(2 - \varepsilon)^2}.$$

This implies $g(X) = 1$.

Conversely, let X be (UNS) and define $\bar{\varepsilon}$ through (8).

By using (6) and (7) we obtain $g(X) = 1 + \delta_1(1, X) = 1 + \dfrac{\delta(\bar{\varepsilon})}{1 - \delta(\bar{\varepsilon})} =$

$= \dfrac{2}{\bar{\varepsilon}}$. The estimate $g(X) \leq \sqrt{2}$ follows then from (3).

PROPOSITION 2.2. *If* X *is* (UNS), *then we have* $G'(X) = \dfrac{2}{g(X)} \geq \sqrt{2}$.

Proof. Take x, y in S such that $\| x - y \| = \varepsilon$; we have $\| x + y \| \leq \leq 2(1 - \delta(\varepsilon))$. Also, for any ε, $\sup \{ \| x + y \| : \| x - y \| = \varepsilon \} = 2(1 - \delta(\varepsilon))$; $\sup\limits_{\| x - y \| = \varepsilon} \min \{ \| x + y \|, \| x - y \| \} = \min(\varepsilon, 2(1 - \delta(\varepsilon)))$. Since $2(1 + - \delta(\varepsilon))$ is decreasing with ε and less than 2 for $\varepsilon = 2$, this implies $G'(X) = = \sup\limits_{x, y \in S} \{ \min(\| x - y \|, \| x + y \|) \} = \sup\limits_{\varepsilon \in [0,2]} (\min(\varepsilon, 2(1 - \delta(\varepsilon)))) = \bar{\varepsilon}$.
By Proposition 2.1 the proof is complete.

REMARK 2.3. It is almost trivial to see that X is (UNS) if and only if $G'(X) < 2$. For that reason, we have $G'(X) \cdot g(X) = 2$ for any X. Also, it is easily seen that

$$G'(X) \leq \rho_X(1) + 1. \tag{9}$$

REMARK 2.4. By using (4), it is possible to obtain lower (upper) estimates of $g(X)$ $(G'(X))$ in terms of $\rho_{X'}(1)$. For example, for any X, from $G'(X) = = \bar{\varepsilon}$ and $\bar{\varepsilon} - 2\delta_X(\bar{\varepsilon}) = 2\bar{\varepsilon} - 2 \leq 2\rho_X(1)$ we obtain $G'(X) \leq \rho_{X'}(1) + 1$.

REMARK 2.5. The inequalities $g(X) \leq \sqrt{2} \leq G'(X)$ for any X are also easy consequences of Dvoretzki theorem (see e.g. [3]). In fact, whenever Y is a subspace of X, we have

$$g(X) \leq g(Y) ; G'(Y) \leq G'(X) . \tag{10}$$

Similar inequalities do not hold in general for $g'(X)$ and $G(X)$.

REMARK 2.6. Let $\bar{\varepsilon} = \sqrt{2}$, i.e., $g(X) = G'(X) = \sqrt{2}$; this implies also $g'(X) = G(X) = \sqrt{2}$. This means that, for any pair x, y in S, we have $\min(\|x - y\|, \|x + y\|) \leq \sqrt{2} \leq \max(\|x - y\|, \|x + y\|)$.

Also, given $\varepsilon > 0$, for any $x(y)$ in S there exists $y(x)$ in S such that $\varepsilon + \min(\|x + y\|, \|x - y\|) \geq \sqrt{2} \geq \max(\|x + y\|, \|x - y\|) - \varepsilon$.

Probably this is true only in Hilbert spaces. This conjecture was raised also in [8]. A related (stronger) conjecture is the following (see [10], p. 87): let x, y in S, $\|x + y\| = \|x - y\|$, imply $\|x + y\| = \sqrt{2}$. Does this force X to be a Hilbert space?

A similar question was raised in § 5 of [1].

3. ESTIMATES IN l_p $(1 < p < +\infty)$

We shall denote by $\{e_i\}$ the elements of the natural basis in these spaces.

PROPOSITION 3.1. *We have* $g(l_p) = \min(2^{1/p}, 2^{1-1/p})$ *and* $G'(l_p) = \max(2^{1/p}, 2^{1-1/p})$.

Proof. It is an immediate consequence of Propositions 2.1 and 2.2, taking into account the well-known values of $\delta_{l_p}(\varepsilon)$ (see [6]).

PROPOSITION 3.2. *We have* $g'(l_p) = G(l_p) = 2^{1/p}$.

Proof. Let $y = \sum_{i=1}^{\infty} a_i e_i \in S(l_p)$. We have $\|e_1 \pm y\| = |1 \pm a_1|^p + (\sum_{i=2}^{\infty} |a_i|^p)^{1/p} = (|1 \pm a_1|^p - |a_1|^p + 1)^{1/p}$. This implies $\min(\|e_1 + y\|, \|e_1 - y\|) = (|1 - |a_1||^p - |a_1|^p + 1)^{1/p} \leq 2^{1/p}$. Now take any $x = \sum_{i=1}^{\infty} \alpha_i e_i \in S(l_p)$. Given any $\varepsilon > 0$, we can choose k such that $|\alpha_k| < \varepsilon$. From $(1 - |\alpha|)^p \geq 1 - p|\alpha|$ $(p \geq 1, |\alpha| \leq 1)$ we obtain: $\min(\|x - e_k\|, \|x + e_k\|) = (1 - |\alpha_k|^p + 1 - |\alpha_k|^p)^{1/p} \geq (1 - p|\alpha_k| + 1 - |\alpha_k|^p)^{1/p} = (2 - \delta)^{1/p}$ with $\delta \to 0$ if $\varepsilon \to 0$. So $\sup_{y \in S(l_p)} |\min(\|x - y\|, \|x + y\|)| \geq 2^{1/p}$ for every $x \in S(l_p)$, and then $g'(l_p) \geq 2^{1/p}$, which implies $g'(l_p) = 2^{1/p}$ because of the former inequality.

With the same notations as above, from $(1 + \alpha)^p \geq 1 + \alpha^p$ $(p \geq 1, \alpha \geq 0)$ we obtain: $\max(\|e_1 + y\|, \|e_1 - y\|) = ((1 + |a_1|)^p - |a_1|^p + 1)^{1/p} \geq 2^{1/p}$, so $G(l_p) \geq 2^{1/p}$. Now, let $x \in S(l^p)$; given $\varepsilon > 0$, take k as before. We have $\max(\| x - e_k \|, \| x + e_k \|) \leq (\sum_{i \neq k} | \alpha_i |^p + (1 + | \alpha_k |^p))^{1/p} \leq (1 + (1 + \varepsilon)^p)^{1/p}$ $= (2 + \delta)^{1/p}$ with $\delta \to 0$ when $\varepsilon \to 0$.

Therefore, for every $x \in S(l_p)$, $\inf_{y \in S(l_p)} \{ \max(\| x - y \|, \| x + y \| \} \leq 2^{1/p}$, which implies $G(l_p) = 2^{1/p}$ because of the former inequality.

REMARK 3.3. It seems to be more difficult to give the exact values of $G(l_p^n)$ and $g'(l_p^n)$. By the proof of Proposition 3.2 it is possible to obtain a lower bound for $g'(l_p^n)$: for example, for any $x = \sum_{i=1}^{n} \alpha_i e_i \in S(l_p^n)$ there exists $k \in N$ such that $| \alpha_k | \leq n^{-1/p}$. This implies $g'(l_p^n) \geq \min(\| x - e_k \|, \| x + e_k \|) \geq$ $\geq (2 - \frac{1}{n} - \frac{p}{1/p})^{1/p}$. Note that $g(l_p^n) = g(l_p)$ and $G'(l_p^n) = G'(l_p)$ for the results of Section 2.

REMARK 3.4. Let $p_X = \sup \{p : X$ is of type $p\}$ and $q_X = \inf \{q : X$ is of cotype $q\}$ (see § 5 of [3] for the definitions). By a theorem of Maurey and Pisier (see [3], p. 77), l_{p_X} and l_{q_X} are both finitely representable in X. So, by using Remark 2.5 and Proposition 3.1, we obtain the following inequality: $g(X) \leq \min(2^{1/q_X}, 2^{1-1/p_X})$. In the same way, $G'(X) \geq \max(2^{1/p_X}, 2^{1-1/q_X})$. In particular, $g(X) = G'(X) = \sqrt{2}$ implies $p_X = q_X = 2$ (compare with Example 5.4 in [3]).

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