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On non-Uniqueness of Complex Geodesies in Convex Bounded Domains


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RIASSUNTO. — Si studiano « combinazioni convesse complesse » per mappe olo­morfe dal disco unità di C in un dominio convesso limitato D di uno spazio di Banach complesso E, e se ne traggono conseguenze sul carattere globale della non unicità per le geodetiche complesse di D.

INTRODUCTION

Let D be a domain of the complex Banach space E. A complex geodesic (for the Carathéodory or for the Kobayashi pseudo-distance) of D is a holo­morphic map of the open unit disc Δ of C into D which is an isometry with re­spect to the Poincaré distance on Δ and the Carathéodory or Kobayashi pseudo­distance on D. [9].

The study of the family of all complex geodesics of D is useful in several questions of complex geometry, for example in the investigation of the group of all holomorphic automorphisms of D (cf. e.g. [8], [12], [13]).

The study of the relationship between the shape of the boundary of D and the existence and uniqueness of the complex geodesics joining two given points of D has produced several results, some of which are the following:

- If D is bounded and convex (in the real sense) and if E is (for example) reflexive, then for any two given points of D there exists at least one complex geodesic containing them in its range, (cf. e.g. [1], [6], [5]).

- If D is strictly convex, all the complex geodesics joining two given points of D have the same range, [13] (and therefore they are the same up to Moebius transformations, [11] Lemma 3.3).

- If D is convex and balanced and μ is the associated seminorm, then for μ(x) > 0 (x ∈ D), the map ζ ↦ ζ x μ(x) is the unique complex geodesic (up to a Moebius transformation) joining 0 and x, if, and only if, x μ(x) is a com­plex extreme point of D, [11].

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Both the study of the existence and the study of the uniqueness of complex geodesies in a given domain D are of interest for the applications.

In [3] we have investigated the problem of non-uniqueness of complex geodesies in the case of a convex, balanced domain $D \subset E$.

In this work, using techniques of $H^p$ spaces, we obtain results of « global » non-uniqueness for complex geodesies in the case of a convex, bounded domain $D$. In particular we prove the following result, which is analogous—in the new hypotheses and for all geodesics of $D$—to that proved in [3], Corollary 2.4:

**Theorem.** Let $D$ be a convex bounded domain in $E$. If $f : \Delta \to D$ is a complex geodesic of $D$ and if $f(\Delta) = \Gamma$ is its range, then the following facts are equivalent:

a) There exist $x_0, y_0 \in \Gamma$ so that the complex geodesic of $D$ joining $x_0$ and $y_0$ is not unique.

b) There exists $x_0 \in \Gamma$ so that the complex geodetic of $D$ containing $x_0$ in its range and tangent to $\Gamma$ at $x_0$ is not unique.

c) For all $x, y \in \Gamma$, the complex geodesic of $D$ joining $x$ and $y$ is not unique.

d) For all $x \in \Gamma$, the complex geodesic of $D$ containing $x$ in its range and tangent to $\Gamma$ at $x$ is not unique.

The above result is a consequence of a more general fact (see Theorem 4) that follows from a « convex complex combination » property of the holomorphic maps from $\Delta$ into $D$, proved in section 2 (Theorem 1).

1. **Preliminaries**

Let $E, F$ be complex Banach spaces and let $D_1, D_2$ be domains in $E, F$ respectively. A holomorphic map $F : D_1 \to D_2$ is a continuous Gateaux analytic map of $D_1$ into $D_2$ (cf. e.g. [2]); the set of all holomorphic maps from $D_1$ into $D_2$ will be denoted by $\text{Hol} (D_1, D_2)$.

For $p = 1, 2, 3, \ldots, \infty$ and $\Delta = \{ \xi \in C : | \xi | < 1 \}$, the symbol $H^p (\Delta)$ will denote, as usual, the Hardy space of all holomorphic functions $f \in \text{Hol} (\Delta, C)$ so that $\| f \|_p < \infty$.

If $D$ is a domain in $E$, $K_D$ and $C_D$ will be, respectively, the Kobayashi and Carathéodory pseudo-distances and $k_D$, $\gamma_D$ the Kobayashi and Carathéodory pseudo-metrics associated with $D$ (cf. e.g. [2]). In the case in which $D$ is convex it turns out that

$$C_D = K_D, \quad k_D = \gamma_D$$

(see e.g. [1], [5]).
Let $f : \Delta \to D$ be a holomorphic map. It can be proved (cf. e.g. [11]) that $f$ is a complex geodesic for $C_D$ if there exist $\xi_0 \neq \xi_1 \in \Delta$ so that

$$C_D(f(\xi_0), f(\xi_1)) = \omega(\xi_0, \xi_1),$$

or if there exists $\xi_0 \in \Delta$ so that

$$\gamma_D(f(\xi_0) ; f'(\xi_0)) = \{1\}_{\xi_0},$$

where $\omega$ and $\{ \}$ denote the Poincaré distance and the Poincaré differential metric of $\Delta$.

Since any two complex geodesics $f$ and $g$ (for $C_D$ or $K_D$) have the same range if, and only if, there exists a Moebius transformation $m$ of $\Delta$ so that $f = g \circ m$, [11], then the complex geodesics $f$ and $g$ will be said to be equal if they differ by a Moebius transformation. In the following uniqueness or non-uniqueness of complex geodesics will always be « up to parametrization ».

2. Convex complex combinations

The following results state some « convex complex combination » properties for the elements of $\text{Hol}(\Delta, D)$, in the case in which $D$ is bounded and convex. Its consequences will be used later in the study of complex geodesics of $D$.

**Theorem 1.** If $D$ is a convex bounded domain in $E$, let $f : \Delta \to D$ and $g : \Delta \to D$ be holomorphic maps. Let $h : \Delta \to \mathbb{C}$ be a function, meromorphic on $\Delta$, with a finite number of poles $\{\alpha_1, \ldots, \alpha_p\}$, continuous on $\Delta \setminus \{\alpha_1, \ldots, \alpha_p\}$ and such that

$$h(e^{i\theta}) \in \mathbb{R}$$

$$0 \leq h(e^{i\theta}) \leq 1$$

for all $\theta \in [0, 2\pi]$.

If the map

$$\chi(\xi) = h(\xi)f(\xi) + (1 - h(\xi))g(\xi) \quad (\xi \in \Delta)$$

is holomorphic on $\Delta$, then $\chi(\Delta) \subseteq D$. If, moreover, $h$ has a zero on $\Delta$, then $\chi(\Delta) \subseteq D$.

**Proof.** Suppose by contradiction that there exists $\xi_0 \in \Delta$ so that $\chi(\xi_0) \in \overline{D}$. Then there exist a continuous linear functional on $E$, $\Lambda$, and a real number $\lambda$ so that

$$\text{Re } \Lambda(\chi) < \lambda < \text{Re } \Lambda(\chi(\xi_0)),$$

which is a contradiction.
for all $x \in \bar{D}$. By definition the function $\Lambda \circ \chi$ is holomorphic (cf. e.g. [2]) and, for any $\theta \in [0, 2\pi]$, we have

$$
\lim_{r \to 1} \Lambda (\chi (re^{i\theta})) = (\Lambda \circ \chi)^* (0) = \\
= \lim_{r \to 1} \Lambda (h (re^{i\theta}) f (re^{i\theta}) + (1 - h (re^{i\theta})) g (re^{i\theta})) = \\
= \lim_{r \to 1} h (re^{i\theta}) \Lambda (f (re^{i\theta})) + (1 - h (re^{i\theta})) \Lambda (g (re^{i\theta})) = \\
= h (ei\theta) \lim_{r \to 1} \Lambda (f (re^{i\theta})) + (1 - h (ei\theta)) \lim_{r \to 1} \Lambda (g (re^{i\theta})).
$$

Since $\bar{D}$ is bounded, the functions $\Lambda \circ f$ and $\Lambda \circ g$ belong to $H^\infty (\Delta)$ and admit radial limits

$$
\lim_{r \to 1} \Lambda (f (re^{i\theta})) = (\Lambda \circ f)^* (0)
$$

and

$$
\lim_{r \to 1} \Lambda (g (re^{i\theta})) = (\Lambda \circ g)^* (0)
$$

for almost all $\theta \in [0, 2\pi]$.

By the assumptions on $h$ and by (1) we can conclude that

$$
(2) \quad \text{Re} (\Lambda \circ \chi)^* (0) = \\
= \text{Re} \{ h (ei\theta) (\Lambda \circ f)^* (0) + (1 - h (ei\theta)) (\Lambda \circ g)^* (0) \} \leq \\
\leq h (ei\theta) \lambda + (1 - h (ei\theta)) \lambda = \lambda.
$$

The map $\Lambda \circ \chi$ belongs to $H^1 (\Delta)$. In fact there exists $t_0 \in (0, 1)$ so that $h$ is continuous on $\Delta \setminus t_0 \Delta$ and that

$$
\lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} | \Lambda \circ \chi (re^{i\theta}) | d\theta = \\
= \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} | h (re^{i\theta}) \Lambda (f (re^{i\theta})) + (1 - h (re^{i\theta})) \Lambda (g (re^{i\theta})) | d\theta \leq \\
= \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| h (re^{i\theta}) \right| | \Lambda (f (re^{i\theta})) | d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} | 1 - h (re^{i\theta}) | \left\| \Lambda (g (re^{i\theta})) \right\| d\theta \\
\leq \max_{\xi \in \Delta \setminus t_0 \Delta} \{ | h (\xi) | \} \| \Lambda \circ f \|_\infty + (1 + \max_{\xi \in \Delta \setminus t_0 \Delta} \{ | h (\xi) | \}) \| \Lambda \circ g \|_\infty < \infty
$$

(see section 1).
It is well known that every map in $H^1(\Delta)$ is the Poisson integral of its radial limit. Hence

$$
\Lambda \circ \chi(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Lambda \circ \chi)^*(t) P_r(\theta - t) \, dt,
$$

where $P_r(\theta - t) = (1 - r^2) / (1 - 2r \cos(\theta - t) + r^2)$.

By (2) we have

$$(3) \quad \text{Re} \left( \Lambda (\chi(re^{i\theta})) \right) =
$$

$$
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Re} ((\Lambda \circ \chi)^*(t)) P_r(\theta - t) \, dt \leq
$$

$$
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda P_r(\theta - t) \, dt \leq \lambda,
$$

and, for $re^{i\theta} = \xi_0$, inequality (3) contradicts formula (1). Hence $\chi(\Delta) \subset D$. Moreover, if $\chi(\Delta) \not\subset \partial D$, then $\chi(\Delta) \cap \partial D = \emptyset$, and the last assertion follows (see e.g. [9]).

We will now apply Theorem 1 to a particular situation.

Let us consider the meromorphic function defined on $C$ by

$$(4) \quad h(\xi) = a\xi + r + \bar{a} \xi, \quad (a \in C \setminus \{0\}, \, r \geq 0)$$

having a simple pole at $0 \in C$.

For $z = re^{i\theta} \in \partial \Delta$ we have

$$(5) \quad h(re^{i\theta}) = ae^{i\theta} + r + \bar{a}e^{-i\theta} = r + 2\text{Re}(ae^{i\theta}),$$

i.e. the function $h(e^{i\theta})$ is real and such that

$$r - 2|a| \leq h(e^{i\theta}) \leq r + 2|a|$$

for all $\theta \in \mathbb{R}$.

The zeros of the function $h$ are exactly the roots $\xi_1$ and $\xi_2$ of the equation $a\xi^2 + r\xi + \bar{a} = 0$. Obviously $\xi_1 \cdot \xi_2 = \bar{a} / a \in \partial \Delta$ and, for $r^2 > 4 |a|^2$ (i.e. $r > 2|a|$ with our assumptions) we have (up to a permutation) $\xi_1 \in \Delta$ (and $\xi_2 \in \overline{\Delta}$). Moreover $\xi_1 \cdot \overline{\xi_2} = 1$. On the other hand, given any $x \in \Delta \setminus \{0\}$, the function $h_a^0$ defined by

$$h_a^0(\xi) = -\bar{a}(\xi - x)(\xi - 1/\bar{a})/((\xi + 1)^2) =$$

$$= (-\bar{a}\xi + (|x|^2 + 1) - \bar{a} / \xi^2) / (|x|^2 + 1)^2$$
is of type (4), real on the boundary of $\Delta$ and such that

$$0 < h_\alpha^0(e^{i\theta}) \leq 1$$

for all $\theta \in [0, 2\pi]$. The function $h_\alpha^0$ has a simple zero at $z$ and a simple pole at 0 on $\Delta$.

If $m$ is a Moebius transformation of $\Delta$, the function

$$h_\alpha^0 \circ m$$

is real on the boundary of $\Delta$, so that

$$0 < h_\alpha^0(m(e^{i\theta})) \leq 1$$

and has a simple zero at $m^{-1}(z)$ and a simple pole at $m^{-1}(0)$. Therefore

**Lemma 2.** For any two points $\gamma \neq \delta \in \Delta$, there exist $t \in (0, 1)$ and a Moebius transformation $m$ so that the holomorphic map defined on $\Delta$ by (see (6)):

$$h_\gamma^t = h_\alpha^0 \circ m$$

has a simple zero at $\gamma$ and a simple pole at $\delta$.

The map $h_\gamma^t$ is real on $\partial \Delta$ and such that

$$0 < h_\gamma^t(e^{i\theta}) \leq 1$$

for all $\theta \in [0, 2\pi]$.

We will conclude this section by stating the following direct consequence of Theorem 1 and Lemma 2.

**Theorem 3.** If $D$ is a convex bounded domain in $\mathbb{E}$, let $f : \Delta \to D$ and $g : \Delta \to D$ be holomorphic maps. For $n \in \mathbb{N}$, $n \geq 1$, let $[\delta_1, \ldots, \delta_n]$ be the set of all the zeroes of the map $(g - f) : \Delta \to \mathbb{E}$, repeated according to their multiplicities. Then for all $m \leq n$ and all $\gamma_1, \ldots, \gamma_m \in \Delta$ (not necessarily distinct) and for all subsets $[\delta_{i_1}, \ldots, \delta_{i_m}]$ of $[\delta_1, \ldots, \delta_n]$, the map

$$\chi = f + h_{\gamma_1}^{\delta_{i_1}} h_{\gamma_2}^{\delta_{i_2}} \ldots h_{\gamma_m}^{\delta_{i_m}} (g - f)$$

is holomorphic and such that:

i) $\chi(\Delta) \subset D$;

ii) the set $[\gamma_1, \ldots, \gamma_m, \delta_1, \ldots, \delta_n] \setminus [\delta_{i_1}, \ldots, \delta_{i_m}]$ is the set of all zeros of the map $(\chi - f) : \Delta \to \mathbb{E}$, repeated according to their multiplicities.

**Proof.** By Lemma 2, the function

$$h(\xi) = h_{\gamma_1}^{\delta_{i_1}}(\xi) h_{\gamma_2}^{\delta_{i_2}}(\xi) \ldots h_{\gamma_m}^{\delta_{i_m}}(\xi) \quad (\xi \in \Delta)$$
satisfies the hypotheses of Theorem 1 and the map

\[ \chi(\xi) = f(\xi) + h(\xi)(g(\xi) - f(\xi)) \quad (\xi \in \Delta) \]

is holomorphic, since \([\delta_{s_1}, \ldots, \delta_{s_m}] \subset [\delta_1, \ldots, \delta_n]\).

Therefore (since, for example, \(h(\gamma_1) = 0\)) Theorem 1 yields that \(\chi(\Delta) \subset D\).

The proof of ii) is obvious. \(\square\)

3. NON-UNIQUENESS OF COMPLEX GEODESICS OF D.

We will now apply the results obtained in section 2 to investigate the character of non-uniqueness for complex geodesics of the convex bounded domain \(D \subset E\).

Remark that if \(f : \Delta \to D\) is a complex geodesic of \(D\) and if \(g : \Delta \to D\) is a holomorphic map so that \((g - f) : \Delta \to E\) has at least two zeros (or one double zero) then (see Section 1 and [11]) \(g\) is a complex geodesic of \(D\), and the assertion of Theorem 3 holds with \(\chi\) complex geodesic of \(D\). In particular we get the following theorem, which explains the meaning of non-uniqueness for complex geodesics in \(D\):

**Theorem 4.** If \(D \subset E\) is a convex bounded domain, let \(f : \Delta \to D\) be a complex geodesic of \(D\) and let \(m \geq 2\) be a natural number. Then the following facts are equivalent:

1) There exist \(p_0 \in \mathbb{N}\), two \(p_0\)-tuples \((\xi^{p_1}_0, \ldots, \xi^{p_0}_0) \in \Delta^{p_0}\), \((n^1_0, \ldots, n^{p_0}_0) \in \mathbb{N}^{p_0}\) with \(n_0^1 + \ldots + n_0^{p_0} = m\), and there exists a complex geodesic \(g : \Delta \to D\) so that \(g(\Delta) \neq f(\Delta)\) and that \(\frac{\partial}{\partial \xi^j}(f - g)(\xi^j_0) = 0\) for all \(j = 1, \ldots, n^i_0\) and all \(i = 1, \ldots, p_0\).

2) For all \(p \in \mathbb{N}\), all \((\xi^1, \ldots, \xi^p) \in \Delta^p\) and all \((n^1, \ldots, n^p) \in \mathbb{N}^p\) with \(n^1 + \ldots + n^p = m\), there exists a complex geodesic \(g : \Delta \to D\) so that \(g(\Delta) \neq f(\Delta)\) and that \(\frac{\partial}{\partial \xi^j}(f - g)(\xi^j) = 0\) for all \(j = 1, \ldots, n^i\) and all \(i = 1, \ldots, p\).

The result announced in the introduction follows by taking \(m = 2\) in Theorem 4. More precisely:

**Corollary 5.** Let \(D \subset E\) be a convex bounded domain and let \(f : \Delta \to D\) be a complex geodesic of \(D\). Then the following facts are equivalent:

a) There exist \(\xi_0 \neq \xi_1 \in \Delta\) so that \(f\) is not the unique complex geodesic joining \(f(\xi_0)\) and \(f(\xi_1)\).

b) There exists \(\xi_2 \in \Delta\) so that \(f\) is not the unique complex geodesic tangent to \(f'(\xi_2)\) at the point \(f(\xi_2)\).
c) For all $\xi \neq \eta \in \Delta$, $f$ is not the unique complex geodesic of $D$ joining $f(\xi)$ and $f(\eta)$.

d) For all $\xi \in \Delta$, $f$ is not the unique complex geodesic tangent to $f'(\xi)$ at the point $f(\xi)$.

REFERENCES