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Graziano Gentili

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Analisi complessa. — *On non-Uniqueness of Complex Geodesics in Convex Bounded Domains.* Nota di GRAZIANO GENTILI (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si studiano « combinazioni convesse complesse » per mappe olomorfe dal disco unità di **C** in un dominio convesso limitato D di uno spazio di Banach complesso E, e se ne traggono conseguenze sul carattere globale della non unicità per le geodetiche complesse di D.

INTRODUCTION

Let D be a domain of the complex Banach space E. A complex geodesic (for the Carathéodory or for the Kobayashi pseudo-distance) of D is a holomorphic map of the open unit disc Δ of **C** into D which is an isometry with respect to the Poincaré distance on Δ and the Carathéodory or Kobayashi pseudo-distance on D, [9].

The study of the family of all complex geodesics of D is useful in several questions of complex geometry, for example in the investigation of the group of all holomorphic automorphisms of D (cf. e.g. [8], [12], [13]).

The study of the relationship between the shape of the boundary of D and the existence and uniqueness of the complex geodesics joining two given points of D has produced several results, some of which are the following:

– If D is bounded and convex (in the real sense) and if E is (for example) reflexive, then for any two given points of D there exists at least one complex geodesic containing them in its range, (cf. e.g. [1], [6], [5]).

– If D is strictly convex, all the complex geodesics joining two given points of D have the same range, [13] (and therefore they are the same up to Moebius transformations, [11] Lemma 3.3).

– If D is convex and balanced and μ is the associated seminorm, then for $\mu(x) > 0$ $(x \in D)$, the map $\zeta \mapsto \zeta \dfrac{x}{\mu(x)}$ is the unique complex geodesic (up to a Moebius transformation) joining 0 and x, if, and only if, $\dfrac{x}{\mu(x)}$ is a complex extreme point of D, [11].

(*) Scuola Normale Superiore, Piazza dei Cavalieri, 7 – 56100 Pisa.
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Both the study of the existence and the study of the uniqueness of complex geodesics in a given domain D are of interest for the applications.

In [3] we have investigated the problem of non-uniqueness of complex geodesics in the case of a convex, *balanced* domain $D \subset E$.

In this work, using techniques of H^p spaces, we obtain results of « global » non-uniqueness for complex geodesics in the case of a convex, *bounded* domain D. In particular we prove the following result, which is analogous—in the new hypotheses and for all geodesics of D—to that proved in [3], Corollary 2.4:

THEOREM. *Let* D *be a convex bounded domain in* E. *If* $f : \Delta \to D$ *is a complex geodesic of* D *and if* $f(\Delta) = \Gamma$ *is its range, then the following facts are equivalent*:

a) *There exist* $x_0, y_0 \in \Gamma$ *so that the complex geodesic of* D *joining* x_0 *and* y_0 *is not unique.*

g) *There exists* $x_0 \in \Gamma$ *so that the complex geodetic of* D *containing* x_0 *in its range and tangent to* Γ *at* x_0 *it not unique.*

c) *For all* $x, y \in \Gamma$, *the complex geodesic of* D *joining* x *and* y *is not unique.*

d) *For all* $x \in \Gamma$, *the complex geodesic of* D *containing* x *in its range and tangent to* Γ *at* x *is not unique.*

The above result is a consequence of a more general fact (see Theorem 4) that follows from a « convex complex combination » property of the holomorphic maps from Δ into D, proved in section 2 (Theorem 1).

1. PRELIMINARIES

Let E, F be complex Banach spaces and let D_1, D_2 be domains in E, F respectively. A *holomorphic map* $F : D_1 \to D_2$ is a continuous Gateaux analytic map of D_1 into D_2 (cf. e.g. [2]); the set of all holomorphic maps from D_1 into D_2 will be denoted by Hol (D_1, D_2).

For $p = 1, 2, 3, \ldots, \infty$ and $\Delta = \{\xi \in \mathbf{C} : |\xi| < 1\}$, the symbol $H(\Delta)^p$ will denote, as usual, the *Hardy space* of all holomorphic functions $f \in$ Hol $(\Delta, \mathbf{C})$ so that $\|f\|_p < \infty$.

If D is a domain in E, K_D and C_D will be, respectively, the Kobayashi and Carathéodory pseudo-distances and k_D, γ_D the Kobayashi and Carathéodory pseudo-metrics associated with D (cf. e.g. [2]). In the case in which D is convex it turns out that

$$C_D \equiv K_D \quad , \quad k_D \equiv \gamma_D$$

(see e.g. [1], [5]).

Let $f : \Delta \to D$ be a holomorphic map. It can be proved (cf. e.g. [11]) that f is a complex geodesic for C_D if there exist $\xi_0 \neq \xi_1 \in \Delta$ so that

$$C_D(f(\xi_0), f(\xi_1)) = \omega(\xi_0, \xi_1),$$

or if there exists $\xi_0 \in \Delta$ so that

$$\gamma_D(f(\xi_0); f'(\xi_0)) = \langle 1 \rangle_{\xi_0},$$

where ω and $\langle \ \rangle$ denote the Poincaré distance and the Poincaré differential metric of Δ.

Since any two complex geodesics f and g (for C_D or K_D) have the same range if, and only if, there exists a Moebius transformation m of Δ so that $f = g \circ m$, [11], then the complex geodesics f and g will be said to be equal if they differ by a Moebius transformation. In the following uniqueness or non-uniqueness of complex geodesics will always be « up to parametrization ».

2. Convex complex combinations

The following results state some « convex complex combination » properties for the elements of Hol (Δ, D), in the case in which D is bounded and convex. Its consequences will be used later in the study of complex geodesics of D.

THEOREM 1. *If* D *is a convex bounded domain in* E, *let* $f : \Delta \to D$ *and* $g : \Delta \to D$ *be holomorphic maps. Let* $h : \bar{\Delta} \to \mathbf{C}$ *be a function, meromorphic on* Δ, *with a finite number of poles* $\{\alpha_1, \ldots, \alpha_p\}$, *continuous on* $\bar{\Delta} \setminus \{\alpha_1, \ldots, \alpha_p\}$ *and such that*

$$h(e^{i\theta}) \in \mathbf{R}$$
$$0 \leq h(e^{i\theta}) \leq 1$$

for all $\theta \in [0, 2\pi]$.

If the map

$$\chi(\xi) = h(\xi) f(\xi) + (1 - h(\xi)) g(\xi) \qquad (\xi \in \Delta)$$

is holomorphic on Δ, *then* $\chi(\Delta) \subset \bar{D}$. *If, moreover,* h *has a zero on* Δ, *then* $\chi(\Delta) \subset D$.

Proof. Suppose by contradiction that there exists $\xi_0 \in \Delta$ so that $\chi(\xi_0) \notin \bar{D}$. Then there exist a continuous linear functional on E, Λ, and a real number λ so that

$$\text{(1)} \qquad \operatorname{Re} \Lambda(x) < \lambda < \operatorname{Re} \Lambda(\chi(\xi_0)),$$

for all $x \in \bar{D}$. By definition the function $\Lambda \circ \chi$ is holomorphic (cf. e.g. [2]) and, for any $\theta \in [0, 2\pi]$, we have

$$\lim_{r\to1} \Lambda(\chi(re^{i\theta})) = (\Lambda \circ \chi)^*(\theta) =$$

$$= \lim_{r\to1} \Lambda(h(re^{i\theta}) f(re^{i\theta}) + (1 - h(re^{i\theta})) g(re^{i\theta})) =$$

$$= \lim_{r\to1} h(re^{i\theta}) \Lambda(f(re^{i\theta})) + (1 - h(re^{i\theta})) \Lambda(g(re^{i\theta})) =$$

$$= h(e^{i\theta}) \lim_{r\to1} \Lambda(f(re^{i\theta})) + (1 - h(e^{i\theta})) \lim_{r\to1} \Lambda(g(re^{i\theta})) .$$

Since $\bar{D}$ is bounded, the functions $\Lambda \circ f$ and $\Lambda \circ g$ belong to $H^\infty(\Delta)$ and admit radial limits

$$\lim_{r\to1} \Lambda(f(re^{i\theta})) = (\Lambda \circ f)^*(\theta)$$

and

$$\lim_{r\to1} \Lambda(g(re^{i\theta})) = (\Lambda \circ g)^*(\theta)$$

for almost all $\theta \in [0, 2\pi]$.

By the assumptions on h and by (1) we can conclude that

$$\text{Re}\,(\Lambda \circ \chi)^*(\theta) = \tag{2}$$

$$= \text{Re}\,\{h(e^{i\theta})(\Lambda \circ f)^*(\theta) + (1 - h(e^{i\theta}))(\Lambda \circ g)^*(\theta)\} \le$$

$$\le h(e^{i\theta})\lambda + (1 - h(e^{i\theta}))\lambda = \lambda .$$

The map $\Lambda \circ \chi$ belongs to $H^1(\Delta)$. In fact there exists $t_0 \in (0, 1)$ so that h is continuous on $\bar{\Delta} \setminus t_0 \Delta$ and that

$$\lim_{r\to1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Lambda \circ \chi(re^{i\theta})\, d\theta| =$$

$$= \lim_{r\to1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(re^{i\theta}) \Lambda(f(re^{i\theta})) + (1 - h(re^{i\theta})) \Lambda(g(re^{i\theta}))|\, d\theta \le$$

$$\le \lim_{r\to1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(re^{i\theta})|\, |\Lambda(f(re^{i\theta}))|\, d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - h(re^{i\theta})\| \Lambda(g(re^{i\theta}))|\, d\theta$$

$$\le \max_{\xi \in \bar{\Delta} \setminus t_0 \Delta} \{|h(\xi)|\} \|\Lambda \circ f\|_\infty + (1 + \max_{\xi \in \bar{\Delta} \setminus t_0 \Delta} \{|h(\xi)|\}) \|\Lambda \circ g\|_\infty < \infty$$

(see section 1).

It is well known that every map in $H^1(\Delta)$ is the Poisson integral of its radial limit. Hence

$$\Lambda \circ \chi (re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Lambda \circ \chi)^* (t) \, P_r (\theta - t) \, dt ,$$

where $P_r(\theta - t) = (1 - r^2) / (1 - 2r \cos(\theta - t) + r^2)$.

By (2) we have

$$\text{(3)} \qquad \operatorname{Re}(\Lambda(\chi(re^{i\theta}))) =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re}((\Lambda \circ \chi)^* (t)) \, P_r(\theta - t) \, dt \leq$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda \, P_r(\theta - t) \, dt \leq \lambda ,$$

and, for $re^{i\theta} = \xi_0$, inequality (3) contradicts formula (1). Hence $\chi(\Delta) \subset \overline{D}$. Moreover, if $\chi(\Delta) \not\subset \partial D$, then $\chi(\Delta) \cap \partial D = \varnothing$, and the last assertion follows (see e.g. [9]). □

We will now apply Theorem 1 to a particular situation.

Let us consider the meromorphic function defined on **C** by

$$\text{(4)} \qquad h(\xi) = a\xi + r + \frac{\overline{a}}{\xi} \qquad (a \in \mathbf{C} \setminus \{0\} , \; r \geq 0)$$

having a simple pole at $0 \in \mathbf{C}$.

For $z = e^{i\theta} \in \partial\Delta$ we have

$$\text{(5)} \qquad h(e^{i\theta}) = ae^{i\theta} + r + \overline{a} e^{-i\theta} = r + 2 \operatorname{Re}(ae^{i\theta}) ,$$

i.e. the function $h(e^{i\theta})$ is real and such that

$$r - 2 |a| \leq h(e^{i\theta}) \leq r + 2 |a|$$

for all $\theta \in R$.

The zeros of the function h are exactly the roots ξ_1 and ξ_2 of the equation $a\xi^2 + r\xi + \overline{a} = 0$. Obviously $\xi_1 \cdot \xi_2 = \overline{a} / a \in \partial\Delta$ and, for $r^2 > 4 |a|^2$ (i.e. $r > 2 |a|$ with our assumptions) we have (up to a permutation) $\xi_1 \in \Delta$ (and $\xi_2 \in \complement \overline{\Delta}$). Moreover $\xi_1 \cdot \overline{\xi}_2 = 1$. On the other hand, given any $\alpha \in \Delta \setminus \{0\}$, the function h_α^0 defined by

$$h_\alpha^0(\xi) = -\overline{\alpha}(\xi - \alpha)(\xi - 1/\overline{\alpha}) / (\xi(|\alpha| + 1)^2) =$$

$$= (-\overline{\alpha}\xi + (|\alpha|^2 + 1) - \frac{\alpha}{\xi}) / (|\alpha| + 1)^2$$

is of type (4), real on the boundary of Δ and such that

$$0 < h_\alpha^0 (e^{i\theta}) \leq 1 \tag{7}$$

for all $\theta \in [0 , 2 \pi]$. The function h_α^0 has a simple zero at α and a simple pole at 0 on Δ.

If m is a Moebius transformation of Δ, the function

$$h_\alpha^0 \circ m$$

is real on the boundary of Δ, so that

$$0 < h_\alpha^0 (m (e^{i\theta})) \leq 1$$

and has a simple zero at $m^{-1} (\alpha)$ and a simple pole at $m^{-1} (0)$. Therefore

LEMMA 2. *For any two points $\gamma \neq \delta \in \Delta$, there exist $t \in (0 , 1)$ and a Moebius transformation m so that the holomorphic map defined on Δ by* (see (6)):

$$h_\gamma^\delta = h_t^0 \circ m$$

has a simple zero at γ and a simple pole at δ.

The map h_γ^δ is real on $\partial\Delta$ and such that

$$0 < h_\gamma^\delta (e^{i\theta}) \leq 1$$

for all $\theta \in [0 , 2 \pi]$.

We will conclude this section by stating the following direct consequence of Theorem 1 and Lemma 2.

THEOREM 3. *If D is a convex bounded domain in E, let $f : \Delta \to D$ and $g : \Delta \to D$ be holomorphic maps. For $n \in \mathbf{N}$, $n \geq 1$, let $[\delta_1, \dots , \delta_n]$ be the set of all the zeroes of the map $(g - f) : \Delta \to E$, repeated according to their multiplicities. Then for all $m \leq n$ and all $\gamma_1 , \dots , \gamma_m \in \Delta$ (not necessarily distinct) and for all subsets $[\delta_{i_1} , \dots , \delta_{i_m}]$ of $[\delta_1 , \dots , \delta_n]$, the map*

$$\chi = f + h_{\gamma_1}^{\delta_{i_1}} h_{\gamma_2}^{\delta_{i_2}} \dots h_{\gamma_m}^{\delta_{i_m}} (g - f)$$

is holomorphic and such that:

i) $\chi (\Delta) \subset D$;

ii) *the set $[\gamma_1 , \dots , \gamma_m, \delta_1 , \dots , \delta_n] \setminus [\delta_{i_1}, \dots , \delta_{i_m}]$ is the set of all zeros of the map $(\chi - f) : \Delta \to E$, repeated according to their multiplicities.*

Proof. By Lemma 2, the function

$$h (\xi) = h_{\gamma_1}^{\delta_{i_1}}(\xi)\, h_{\gamma_2}^{\delta_{i_2}}(\xi) \dots h_{\gamma_m}^{\delta_{i_m}}(\xi) \qquad (\xi \in \Delta)$$

satisfies the hypotheses of Theorem 1 and the map

$$\chi(\xi) = f(\xi) + h(\xi)(g(\xi) - f(\xi)) \qquad (\xi \in \Delta)$$

is holomorphic, since $[\delta_{i_1}, \ldots, \delta_{i_m}] \subset [\delta_1, \ldots, \delta_n]$.

Therefore (since, for example, $h(\gamma_1) = 0$) Theorem 1 yields that $\chi(\Delta) \subset D$. The proof of ii) is obvious. □

3. Non-uniqueness of complex geodesics of D.

We will now apply the results obtained in section 2 to investigate the character of non-uniqueness for complex geodesics of the convex bounded domain $D \subset E$.

Remark that if $f : \Delta \to D$ is a complex geodesic of D and if $g : \Delta \to D$ is a holomorphic map so that $(g - f) : \Delta \to E$ has at least two zeros (or one double zero) then (see Section 1 and [11]) g is a complex geodesic of D, and the assertion of Theorem 3 holds with χ complex geodesic of D. In particular we get the following theorem, which explains the meaning of non-uniqueness for complex geodesics in D:

Theorem 4. *If $D \subset E$ is a convex bounded domain, let $f : \Delta \to D$ be a complex geodesic of D and let $m \geq 2$ be a natural number. Then the following facts are equivalent:*

1) *There exist $p_o \in N$, two p_o-tuples $(\xi_0^1, \ldots, \xi_0^{p_0}) \in \Delta^{p_0}$, $(n_0^1, \ldots, n_0^{p_0}) \in \mathbf{N}^{p_0}$ with $n_0^1 + \ldots + n_0^{p_0} = m$, and there exists a complex geodesic $g : \Delta \to D$ so that $g(\Delta) \neq f(\Delta)$ and that $\frac{\partial}{\partial z^j}(f - g)(\xi_0^i) = 0$ for all $j = 1, \ldots, n_0^i$ and all $i = 1, \ldots, p_0$.*

2) *For all $p \in \mathbf{N}$, all $(\xi^1, \ldots, \xi^p) \in \Delta^p$ and all $(n^1, \ldots, n^p) \in \mathbf{N}^p$ with $n^1 + \ldots + n^p = m$, there exists a complex geodesic $g : \Delta \to D$ so that $g(\Delta) \neq f(\Delta)$ and that $\frac{\partial}{\partial z^j}(f - g)(\xi^i) = 0$ for all $j = 1, \ldots, n^i$ and all $i = 1, \ldots, p$.*

The result announced in the introduction follows by taking $m = 2$ in Theorem 4. More precisely:

Corollary 5. *Let $D \subset E$ be a convex bounded domain and let $f : \Delta \to D$ be a complex geodesic of D. Then the following facts are equivalent:*

a) There exist $\xi_0 \neq \xi_1 \in \Delta$ so that f is not the unique complex geodesic joining $f(\xi_0)$ and $f(\xi_1)$.

b) There exists $\xi_2 \in \Delta$ so that f it not the unique complex geodesic tangent to $f'(\xi_2)$ at the point $f(\xi_2)$.

c) *For all* $\xi \neq \eta \in \Delta$, f *is not the unique complex geodesic of* D *joining* $f(\xi)$ *and* $f(\eta)$.

d) *For all* $\zeta \in \Delta$, f *is not the unique complex geodesic tangent to* $f'(\zeta)$ *at the point* $f(\zeta)$.

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