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Bounded solutions on the Real line to non-autonomous Riccati Equations

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Equazioni differenziali ordinarie. — *Bounded solutions on the Real line to non-autonomous Riccati Equations.* Nota di GIUSEPPE DA PRATO e AKIRA ICHIKAWA, presentata (*) dal Corrisp. R. CONTI.

RIASSUNTO. — Si dà un risultato di esistenza e unicità di una soluzione limitata in $]-\infty, +\infty[$ per un'equazione di Riccati infinito-dimensionale.

Let H be a Hilbert space and $A = \{A(t)\}_{t \in \mathbf{R}}$ a family of linear operators in H. We assume:

(1)
- i) For any $t \in \mathbf{R}$, $A(t)$ generates strongly continuous semi-groups in H.
- ii) There exists an evolution operator $U_A(t, s)$ (resp. $U_{A^*}(t, s)$) with $-\infty < s \leq t < \infty$ attached to the family A (resp. A^*). Moreover $U_A(t, s)\, x = \lim_{n \to \infty} U_{A_n^*}(t, s)\, x$, $U_{A_n^*}(t, s) = \lim_{n \to \infty} U_{A_n^*}(t, s)\, x$, $x \in H$ where $A_n(t)$ (resp. $A_n^*(t)$) is the Yosida approximation of $A(t)$ (resp. $A_n(t)$).
- iii) There exists $M > 0$, $\omega \in \mathbf{R}$ so that $|U_A(t, s)| + |U_{A^*}(t, s)| \leq M e^{\omega(t-s)}$ $-\infty < s \leq t < +\infty$.

Let U be another Hilbert space. Let $B = \{B(t)\}_{t \in \mathbf{R}}$, $M = \{M(t)\}_{t \in \mathbf{R}}$ and $N = \{N(t)\}_{t \in \mathbf{R}}$ be families of linear operators. We assume:

(2)
- i) $B(t) \in L(U; H)$, $\forall t \in \mathbf{R}$ and B is strongly continuous and bounded.
- ii) $M(t) \in L(H)$, $\forall t \in \mathbf{R}$ and M is strongly continuous and bounded.
- iii) $N(t) \in L(U)$, $N(t) \geq \varepsilon > 0$ for some $\varepsilon > 0$ and N is strongly continuous and bounded.

(*) Nella seduta del 22 novembre 1985.

We also assume:

(3)
i) (A , B) is stabilizable, i.e. there exists a mapping $K : \mathbf{R} \to L(H ; U)$ strongly continuous bounded so that U_{A-BK} is stable (i.e. $| U_{A-BK}(t, s) | \leq C e^{-\omega(t-s)}$, $\omega > 0$, $t > s$) where U_{A-BK} is the evolution operator generated by A–BK.

ii) (A , M) is detectable i.e. there exists a mapping $K_1 : \mathbf{R} \to L(H)$ strongly continuous bounded so that $U_{A-K_1 M}$ is stable (i.e. $| U_{A-MK_1}(t, s) | \leq C_1 e^{-\omega_1(t-s)}$, $\omega_1 > 0$, $t > s$).

We are interested in bounded solutions in $]-\infty, \infty[$ of the Riccati equation:

$$Q' + A^*Q + QA = QBN^{-1} B^*Q + M^*M = 0 . \tag{4}$$

We say that Q is a *mild solution* of (4) if we have:

$$Q(t)\, x = U_A^*(s, t)\, Q(s)\, U_A(s, t) + \tag{5}$$

$$+ \int_t^s U_A^*(v, t)\,(M^*(v)\, M(v) - Q(v)\, B(v)\, N^{-1}(v)\, B^*(v)\, Q(v))\, U_A(v, t)\, x\, dv$$

for any $x \in H$, $-\infty < t < s < +\infty$.

PROPOSITION 1. *Assume* (1), (2), (3)–i. *Then there exists a* $Q : \mathbf{R} \to L(H)$ *strongly continuous so that:*

i) $Q(t) \geq 0 \quad \forall t \in \mathbf{R}$;

ii) $\| Q(t) \| \leq C$ *for some constant* C, $\forall t \in \mathbf{R}$;

iii) Q *is a solution to eq.* (4).

Proof. Let $n \in \mathbf{N}$; then there exists a unique positive mild solution Q_n of the problem:

$$\begin{cases} Q_n' + A^*Q_n + Q_nA - Q_nBN^{-1} B^*Q_n + M^*M = 0 \\ Q_n(n) = 0 \qquad t \in]-\infty, n[\end{cases} \tag{6}$$

moreover it holds

$$Q_n(t) \leq Q_{n+1}(t) \quad , \quad t \in]-\infty, n[\, . \tag{7}$$

Let y be the mild solution of the problem:

$$(8) \qquad \begin{cases} y'(s) = A(s)\,y(s) + B(s)\,u(s) & s \in [t, n] \\ y(t) = x & -\infty < t < n \end{cases}$$

where $u \in L^2(t, n; U)$.

We have

$$\frac{d}{ds}\langle Q_n(s)\,y(s), y(s)\rangle = |N^{-1/2}(u + N^{-1}B^*y)(s)|^2 - |My|^2 - |N^{1/2}u|^2$$

which implies

$$(9) \qquad \langle Q_n(t)\,x, x\rangle + \int_t^n |N^{-1/2}(u + N^{-1}B^*y)|^2\,ds = \int_t^n [|My|^2 + |N^{1/2}u|^2]\,ds$$

and hence

$$(10) \qquad \langle Q_n(t)\,x, x\rangle \le \int_t^n [|My|^2 + |N^{1/2}u|^2]\,ds .$$

Let now K be such that (3)–i holds. By taking

$$\begin{cases} u(s) = -K(s)\,y(s) \\ y(s) = U_{A-BK}(s, t)\,x \end{cases}$$

in (10) we have

$$\langle Q_n(t)\,x\rangle \le \frac{(\|M\|^2 C^2 + \|N\|\ \|K\|^2 C^2)\,|x|^2}{\omega}(1 - e^{-\omega(n-t)}) .$$

Thus

$$(11) \qquad |Q_n(t)| \le \frac{c^2}{\omega}(\|M\| + \|N\|\ \|K\|^2) \quad , \quad -\infty < t < n .$$

Since Q_n is increasing in n there exists a limit

$$(12) \qquad Q(t)\,x = \lim_{n\to\infty} Q_n(t)\,x \quad , \quad \forall x \in H .$$

Now the conclusion follows letting n go to infinity in the equality

$$Q_n(t)\,x = U_A^*(s,t)\,Q_n(s)\,U_A(s,t)\,x +$$

$$+\int_t^s U_A^*(v,t)\,(M^*(v)\,M(v) - Q_n\,B(v)\,N^{-1}(v)\,B^*(v)\,Q_n(v))\,U_A(v,t)\,x\,dv\,. \quad \blacksquare$$

PROPOSITION 2. *Assume* (1), (2), (3)-ii. *Then eq.* (4) *has at most one bounded mild solution.*

Proof. Let Q be a bounded solution of eq. (4).

1st step. We shall show that $L = A - BN^{-1}B^*Q$ is stable i.e., U_L is stable. Set $K = BN^{-1}B^*$, then eq. (4) is equivalent to

$$Q' + L^*Q + QL + QKQ + M^*M = 0\,. \tag{13}$$

We have

$$\frac{d}{ds}\langle Q(s)\,U_L(s,t)\,x\,,\,U_L(s,t)\,x\rangle = -\,|\sqrt{K(s)}\,Q(s)\,U_L(s,t)\,x\,|^2$$

$$-\,|\,M(s)\,U_L(s,t)\,x\,|^2\,,$$

which implies

$$\langle Q(t)\,x\,,\,x\rangle = \langle Q(s)\,U_L(s,t)\,x\,,\,U_L(s,t)\,x\rangle + \tag{14}$$

$$+\int_t^s [\,|\sqrt{K(\sigma)}\,Q(\sigma)\,U_L(\sigma,t)\,x\,|^2 + |\,M(\sigma)\,U_L(\sigma,t)\,x\,|^2]\,d\sigma\,.$$

Thus

$$MU_L(\cdot,t)\,X \quad , \quad \sqrt{K}\,QU_L(\cdot,t)\,x \in L^2(t\,,\infty;H)\,. \tag{15}$$

Now, by (3)-ii there exists a K_1 such that $A - K_1M = D$ is stable. From the equality

$$U_L(s,t)\,x = U_D(s,t)\,x - \int_t^s U_D(s,v)\,(K_1M - KQ)\,U_L(v,t)\,x\,dv$$

and (15) it follows $U_L(\cdot,t)\,x \in L^2(t\,,\cdot\,;H)$ and by Datko [3] this implies that L is stable.

2nd step. Now we prove uniqueness.

Let V be another bounded solution of eq. (4). Setting $Z = Q - V$ we have

$$Z' + L^*Z + ZL + ZKZ = 0 .$$

From this we have

$$\frac{d}{ds} \langle Z(s) U_L(s,t)x, Z(s) U_L(s,t)x\rangle = - | \sqrt{K(s)}\, Z(s) U_L(s,t)x|^2$$

which implies

$$\langle Z(t)x, x\rangle = \int_t^s | \sqrt{K(\sigma)}\, Z(\sigma) U_L(\sigma,t)x|^2 d\sigma + \langle Z(s) U_L(s,t)x, U_L(s,t)x\rangle .$$

Thus $\langle Z(t)x, x\rangle \geq \langle Z(s) U_L(s,t)x, U_L(s,t)x\rangle$. Letting s tend to infinity, we get $\langle Z(t)x, x\rangle \geq 0$.

Hence $Q \geq V$. Interchanging Q and V we conclude $Q = V$. ∎

COROLLARY 1. *Assume* (1), (2), (3). *Let* Q *be the unique bounded mild solution of eq.* (4).

i) *If* A, B, M, N *are constant we have*

$$Q(t) = \text{const} = Q_\infty$$

and Q_∞ *is the unique solution of the Algebraic Riccati equation.*

$$A^*Q + QA + M^*M - QBN^{-1}B^*Q = 0 . \tag{16}$$

ii) *If* A, B, M, N *are* T*-periodic then* $Q(t)$ *is the unique* T*-periodic solution of eq.* (4).

Proof. Set $V(t) = Q(t + \lambda)$, $\lambda \in H$; then in the first case V is also a bounded solution of (4), so that by uniqueness

$$Q(t+\lambda) = Q(t) \quad , \quad \forall \lambda \in \mathbf{R} .$$

Set $Q_\infty = Q(t)$, then it is easy to show that Q_∞ satisfies (16).

Consider now the second case. Then $V(t) = Q(t + T)$ is also a solution of (4) and by uniqueness we have

$$Q(t+T) = Q(t) .$$

References

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